THE INNER DERIVATIONS OF A JORDAN ALGEBRA

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A Jordan algebra J is an algebra over a field Φ of characteristic $\neq 2$ with a product $a \cdot b$ satisfying

$$(1) \qquad a \cdot b = b \cdot a,$$

$$(2) \qquad (a^{\cdot 2} \cdot b) \cdot a = a^{\cdot 2} \cdot (b \cdot a)$$

where $a^{\cdot 2} = a \cdot a$. The following operator identity is easily derived from (1) and the linearized form of (2)

$$(3) \qquad [a_R[b_R c_R]] = (a[b_R c_R])_R \qquad \text{for } a, b, c, \in J$$

where x_R denotes right multiplication by x and $[uv] = uv - vu$. Letting $D = [b_R c_R]$, we see that (3) implies $(d \cdot a)D - (dD) \cdot a = d \cdot (aD)$ for $a, d \in J$. In other words, D is a derivation of the Jordan algebra J. Hence every mapping of the form $\sum [b_{iR} c_{iR}]$ is a derivation. We shall call such derivations *inner* derivations and denote the set of all inner derivations of J by Inder(J). It is easily shown that Inder(J) is an ideal in the Lie algebra of all derivations of J. We shall show that if the characteristic of Φ is $p \neq 0$, then Inder(J) is a restricted Lie algebra; that is, $D^p \in$ Inder(J) if $D \in$ Inder(J).

If $\mathfrak{A}$ is an associative algebra, we denote by $\mathfrak{A}^+$ the Jordan algebra whose vector space is that of $\mathfrak{A}$ and whose multiplication is $u \cdot v = \frac{1}{2}(uv + vu)$. A Jordan algebra J is *special* if J is a subalgebra of $\mathfrak{A}^+$ for some associative algebra $\mathfrak{A}$. Let $\Phi\{x_1, \cdots, x_n\}$ be the free associative algebra generated by $x_1, \cdots, x_n$ over the field Φ. An element u in $\Phi\{x_1, \cdots, x_n\}$ is called *Jordan* if u is in the subalgebra of $\Phi\{x_1, \cdots, x_n\}^+$ generated by 1 and $x_1, \cdots, x_n$. We can now state the following

LEMMA. *If Φ is of characteristic $p \neq 0$, 2, then there exist Jordan elements $f_i(x, y)$, $i = 1$, 2 in $\Phi\{x, y\}$ such that $[xy]^p = [x, f_1(x, y)] + [y, f_2(x, y)]$.*

PROOF. We introduce the reversal operation in $\Phi\{x, y\}$ which is an involution $a \rightarrow a^*$ such that $x^* = x$ and $y^* = y$. We say a is reversible if $a^* = a$. Let $\mathfrak{M}$ be the subspace of $\Phi\{x, y\}$ of all elements of the form $[xa] + [yb]$ where a and b are reversible. Since by Cohn's theorem

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[1] every reversible element of $\Phi\{x, y\}$ is a Jordan element, we need only show that $[xy]^p \in \mathfrak{M}$.

Let A be the set consisting of the 2^p monomials of the form $u = a_1 a_2 \cdots a_p$ where $a_i = xy$ or $-yx$, $i = 1, 2, \cdots, p$. We define an equivalence relation $\sim$ on A by $u \sim v$ if $v = a_{1\sigma} a_{2\sigma} \cdots a_{p\sigma}$ where u is as above and σ is a cyclic permutation of $(1, 2, \cdots, p)$. An equivalence class determined by $\sim$ has either 1 or p elements since the cyclic permutations of $(1, 2, \cdots, p)$ form a cyclic group of order p. Let $A_1 = \{u_{11} = (xy)^p\}$, $A_2 = \{u_{21} = (-yx)^p\}$, $A_3 = \{u_{31}, u_{32}, \cdots, u_{3p}\}$, $\cdots$, $A_e = \{u_{e1}, u_{e2}, \cdots, u_{ep}\}$ be the equivalence classes determined by $\sim$.

If $r = b_1 b_2 b_3 \cdots b_{2p}$ and $s = b_2 b_3 \cdots b_{2p} b_1$ where $b_i = x$ or y, $i = 1, 2, 3, \cdots, 2p$, then $(r - r^*) - (s - s^*) = [b_1, b_2 b_3 \cdots b_{2p} + b_{2p} \cdots b_3 b_2] \in \mathfrak{M}$. Thus, if $t = b_{1\tau} b_{2\tau} \cdots b_{(2p)\tau}$ where τ is a cyclic permutation of $(1, 2, \cdots, 2p)$, then $(r - r^*) - (t - t^*) \in \mathfrak{M}$. In particular, $(u - u^*) - (v - v^*) \in \mathfrak{M}$ if $u, v \in A$ and $u \sim v$. Also, $(u_{11} - u_{11}^*) + (u_{21} - u_{21}^*) \in \mathfrak{M}$.

Now we may write $[xy]^p = (xy - yx)^p = \sum u \in Au$. Since $([xy]^p)^* = -[xy]^p$, we have

$$
\begin{aligned}
[xy]^p &= \frac{1}{2} \sum u \in A(u - u^*) \\
&= \frac{1}{2}\left\{u_{11} - \overset{*}{u}_{11} + u_{21} - \overset{*}{u}_{21} + \sum_{i=3}^{e} \sum_{j=1}^{p} (u_{ij} - \overset{*}{u}_{ij})\right\} \\
&= \frac{1}{2}\left\{m + \sum_{i=3}^{e} (p(u_{i1} - \overset{*}{u}_{i1}) + m_i)\right\}
\end{aligned}
$$

where $m, m_i \in \mathfrak{M}$, $i = 3, \cdots, p$. Hence $[xy]^p \in \mathfrak{M}$.

THEOREM. *If the characteristic of Φ is $p \neq 0$, then the Lie algebra* Inder(J) *is restricted.*

PROOF. We recall the following two identities which hold in any associative algebra over Φ [2, pp. 186–187]:

$$u(\operatorname{ad} v)^p = u(\operatorname{ad} v^p), \tag{4}$$

$$(u + v)^p = u^p + v^p + \sum_{i=1}^{p-1} s_i(u, v) \tag{5}$$

where $x(\operatorname{ad} y) = [xy]$ and $s_i(u, v)$ is in the Lie subalgebra generated by u and v. Let $D = \sum [b_{iR} c_{iR}] \in \text{Inder}(J)$. In view of (5), we will have $D^p \in \text{Inder}(J)$ if $[b_R c_R]^p \in \text{Inder}(J)$ for $b, c \in J$.

First we assume that J is special. By writing both sides in terms of the associative multiplication, one verifies the following identity

$$a[b_Rc_R] = (\tfrac{1}{4})[a[bc]] \quad a, b, c \in J. \tag{6}$$

As an immediate consequence of (6) and (4) we have

$$a[b_Rc_R]^p = (\tfrac{1}{4})^p[a[bc]^p] \qquad a, b, c \in J. \tag{7}$$

Using the lemma, we may write

$$[bc]^p = [bf_1(b, c)] + [cf_2(b, c)] \qquad b, c \in J. \tag{8}$$

Combining (7) and (8) and making use of (6), we see

$$a[b_Rc_R]^p = a(\tfrac{1}{4})^{p-1}\{[b_R(f_1(b, c))_R] + [c_R(f_2(b, c))_R]\} \quad a, b, c \in J. \tag{9}$$

Since (9) involves only a, b, and c with a linear and since (9) holds for all special Jordan algebras over Φ, it must hold for all Jordan algebras over Φ by MacDonald's theorem [**3**]. Thus $[b_Rc_R]^p \in \mathrm{Inder}(J)$, and $\mathrm{Inder}(J)$ is restricted.

References

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