THE MATRIX GROUP OF TWO PERMUTATION GROUPS[1]

BY EDGAR M. PALMER AND ROBERT W. ROBINSON

Communicated by M. Kac, September 21, 1966

1. **Introduction.** In his remarkable combinatorial paper, Redfield [3] derived a formula for the number of equivalence classes of "range correspondences." It can be obtained by applying Burnside's well-known theorem [1] for the number of orbits determined by a permutation group and is used to enumerate superposed graphs [2]. In this announcement we construct a more general permutation group, called the "matrix group" and give an explicit expression for the number of orbits it determines. This result enables us to enumerate superposed graphs composed of interchangeable copies of the same graph.

2. **The matrix group.** Let A be a permutation group of degree m acting on the set $X=\{1, 2, \cdots, m\}$. For any permutation α in A, we denote by $j_k(\alpha)$ the number of cycles of length k in the disjoint cycle decomposition of α. Let $a_1, a_2, \cdots, a_m$ be variables. The cycle index $Z(A)$ is given by

$$Z(A) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{m} a_k^{j_k(\alpha)} . \tag{1}$$

It is often convenient to write $Z(A)=Z(A; a_1, a_2, \cdots, a_m)$ to indicate the variables used.

Now let B be another permutation group of degree n acting on $Y=\{1, 2, \cdots, n\}$. Let W be the collection of m by n matrices in which the elements of each row are the n objects in Y. Thus there are $(n!)^m$ matrices in W. Two matrices in W are called *column-equivalent* if one can be obtained from the other by a permutation of the columns. Hence there are $(n!)^{m-1}$ column-equivalence classes.

The *matrix group* of A and B, denoted $[A; B]$, acts on the column-equivalence classes as follows. For each permutation α in A and each sequence $\beta_1, \beta_2, \cdots, \beta_m$ of m permutations with β_i in B, there is a permutation, denoted $[\alpha; \beta_1, \beta_2, \cdots, \beta_m]$ in $[A; B]$ such that the column-equivalence class to which the matrix $[w_{i,j}]$ belongs is sent by $[\alpha; \beta_1, \beta_2, \cdots, \beta_m]$ to the class to which $[\beta_i w_{\alpha i,j}]$ belongs. That is, α first determines a permutation of the rows and then each β_i permutes the entries in the ith row.

[1] Work supported in part by a grant from the U. S. Office of Naval Research.

The number of orbits determined by the matrix group $[A; B]$ is denoted $N[A; B]$. Let G be a graph with n points. The automorphism group of G, denoted $\Gamma(G)$, is a permutation group of degree n. The symmetric group of degree m is denoted, as usual, by S_m. The following theorem is a consequence of the construction of the matrix group.

THEOREM 1. *The number of superposed graphs composed of m interchangeable copies of G is $N[S_m; \Gamma(G)]$.*

3. The enumeration theorem. In order to state the theorem which gives the formula for the number $N[A; B]$ of orbits of the matrix group $[A; B]$, we require several definitions.

Let R be the ring of real polynomials in the variables $b_1, b_2, \cdots, b_n$. First we recall the operation U introduced by Redfield. For any sequence of m monomials $b_1^{j_1}b_2^{j_2} \cdots b_n^{j_n} \cdots b_1^{i_1}b_2^{i_2} \cdots b_n^{i_n}$, we let

$$(2) \qquad (b_1^{j_1}b_2^{j_2} \cdots b_n^{j_n})U \cdots U(b_1^{i_1}b_2^{i_2} \cdots b_n^{i_n}) = b_1^{j_1}b_2^{j_2} \cdots b_n^{j_n}\left(\prod_k j_k!k^{j_k}\right)^{m-1},$$

if $j_k = \cdots = i_k$ for all k, and it is zero otherwise. This operation is now extended linearly to $R \otimes \cdots \otimes R$.

Now for each positive integer r, we define a function J_r: $R \to R$. It is convenient first to define a sequence of functions $d_1, d_2, \cdots$ which depend on the integers r and $k \geqq 1$. For each $i = 1, 2, \cdots$ we let

$$(3) \qquad \begin{aligned} d_i &= b_{ki}/k \qquad \text{if } i \mid r \text{ and } (r/i, k) = 1; \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Using S_j to denote the symmetric group on j objects, we define $J_r(b_k^j)$ by

$$(4) \qquad J_r(b_k^j) = j!k^j Z(S_j; d_1, d_2, \cdots, d_j).$$

Observe that for any prime p, the previous formula (4) for $J_p(b_k^j)$ can be written

$$(5) \qquad \begin{aligned} J_p(b_k^j) &= 0, \qquad \text{if } p \mid k \text{ but } p \nmid j, \\ &= (j!k^{j(p-1)/p} b_{pk}^{j/p})/((j/p)!p^{j/p}), \qquad \text{if } p \mid k \text{ and } p \mid j, \\ &= \sum_{s=0}^{[j/p]} (j!k^{(p-1)s} b_{kp}^s b_k^{j-sp})/((j-sp)!s!p^s), \qquad \text{if } p \nmid k. \end{aligned}$$

For monomials $b_1^{j_1}b_2^{j_2} \cdots b_n^{j_2}$, we define J_r by

$$J_r(b_1^{j_1} b_2^{j_2} \cdots b_n^{j_n}) = \prod_{k=1}^{n} J_r(b_k^{j_k}). \tag{6}$$

Now J_r is extended linearly to R. In particular

$$J_r(Z(B)) = \frac{1}{|B|} \sum_{\beta \in B} J_r\left(\prod_{k=1}^{n} b_k^{j_k(\beta)}\right) \tag{7}$$

and

$$J_1(Z(B)) = Z(B). \tag{8}$$

Next we define a product for the collection of functions $\{J_r\}$. We set

$$J_1^{i_1} J_2^{i_2} \cdots J_m^{i_m}(Z(B)) = J_1^{i_1}(Z(B)) U \cdots U J_m^{i_m}(Z(B)) \tag{9}$$

where it is understood that before evaluating the right member of (9), each $J_r^{i_r}(Z(B))$ is replaced by the U-product of length i_r,

$$J_r(Z(B)) U \cdots U J_r(Z(B)).$$

With the help of this notation, the main result takes the following form. A complete proof will appear elsewhere.

THEOREM 2. *The number of orbits* $N[A; B]$ *determined by the matrix group* $[A; B]$ *is*

$$N[A; B] = [Z(A; J_1, J_2, \cdots, J_m) Z(B)]_{b_i=1}. \tag{10}$$

4. **An application to superposed graphs.** Let P_4 be a path on four points. From Theorem 1, the number of superposed graphs composed of two interchangeable copies of P_4 is $N[S_2; \Gamma(P_4)]$. Applying (8) and (5), we have

$$\begin{aligned} Z(S_2; J_1, J_2) Z(\Gamma(P_4)) &= \tfrac{1}{2}(J_1^2 + J_2)\tfrac{1}{2}(b_1^4 + b_2^2) \\ &= \tfrac{1}{2}\{\tfrac{1}{4}(24 b_1^4 + 8 b_2^2) \\ &\qquad + \tfrac{1}{2}(b_1^4 + 6 b_1^2 b_2 + 3 b_2^2 + 2 b_4)\}. \end{aligned}$$

The coefficient sum of the last expression is 7, which by Theorem 2 is the value of $N[S_2; \Gamma(P_4)]$. This is confirmed by the figure which shows all eight superposed graphs composed of two copies of P_4. Solid and dashed lines are used to distinguish the copies. Interchanging the solid and dashed lines permutes the last two graphs in the figure and leaves each of the first six fixed.

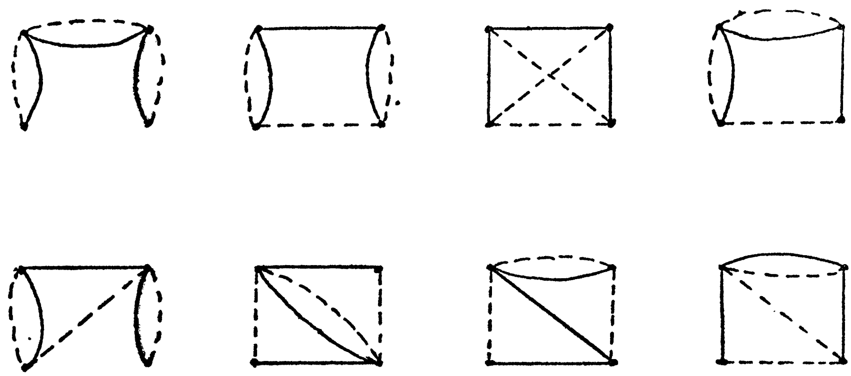

FIGURE 1

REFERENCES

1. W. Burnside, *Theory of groups of finite order*, 2nd ed., Dover, New York, 1955, p. 191.

2. F. Harary, *Combinatorial problems in a graphical enumeration*, pp. 185–217, Applied Combinatorial Mathematics, E. Beckenbach, ed., Wiley, New York, 1964.

3. J. H. Redfield, *The theory of group-reduced distributions*, Amer. J. Math. **49** (1927), 433–455.

THE UNIVERSITY OF MICHIGAN