The formulation and study of this axiomatic system was suggested by the discovery [1] that Khintchine's factorization theorems for the convolution semigroup of probability distributions on R can be extended to the semigroup of renewal sequences, among others.

REFERENCES

1. D. G. Kendall, *Renewal sequences and their arithmetic,* Proc. Loutraki Symposium on Probabilistic Methods in Analysis, Springer, Berlin, (to appear in 1966).

2. ———, *Delphic semi-groups, infinitely divisible regenerative phenomena, and the arithmetic of p-functions,* (in preparation).

CHURCHILL COLLEGE,
CAMBRIDGE, ENGLAND.

AN ALGEBRAIC CONJUGACY INVARIANT FOR MEASURE PRESERVING TRANSFORMATIONS[1]

BY WILLIAM B. ARVESON

Communicated by P. R. Halmos, August 1, 1966

Let T be an invertible, ergodic, measure-preserving transformation of a separable, nonatomic probability space $(X, \mathcal{B}, m)$, and let U be the induced unitary operator acting in $L^2(X, \mathcal{B}, m)$. Let $\mathcal{A}(T)$ be the Banach algebra generated by the multiplication algebra and the nonnegative powers of U. It is shown that, if S is another such transformation, then S and T are conjugate if, and only if, $\mathcal{A}(S)$ and $\mathcal{A}(T)$ are unitarily equivalent. Thus, the conjugacy problem for ergodic transformations is equivalent to multiplicity theory for the algebras $\mathcal{A}(T)$. While much remains to be learned about these operator algebras, similar ones have been studied in [5] and [1]. Finally, $\mathcal{A}(T)$ can be realized concretely as an algebra of operator-valued analytic functions in the unit disc.

In §2 we describe generalizations of the C^*-algebra constructed in §1; it turns out that pathology appears as soon as the group involved fails to be amenable, and only in that case.

Full details and further developments will appear elsewhere.

1. The algebras $\mathcal{A}$ and $\mathcal{B}$. For definiteness, we assume all transformations act on the unit interval, are Borel measurable, and pre-

[1] This research was supported by the U. S. Army Research Office (Durham).

serve Lebesgue measure. Let $\mathcal{H}$ be the Hilbert space $L^2[0, 1]$, and let M be the von Neumann algebra of all multiplications L_f by bounded measurable functions f. Lebesgue measure lifts to a countably additive measure on the projections of M, such that $m(I)=1$, and $m(P)=0$ iff $P=0$. If T is as above, then the map α: $L_f \rightarrow L_{f\circ T}$ is a *-automorphism of M such that $m \circ \alpha(P)=m(P)$ for all projections P, and α is ergodic ($\alpha(P)=P$ implies $P=0$ or $P=I$). Every such α comes from such a T, and the correspondence is essentially 1:1.

Now let α be an ergodic, m-preserving *-automorphism of M, fixed from here on. While the powers α^n ($n\neq 0$) of α need not be ergodic, they are necessarily *freely-acting* in the sense that every nonzero projection in M contains a subprojection $Q_n \neq 0$ such that $\alpha^n(Q_n) \perp Q_n$. If β is another ergodic, m-preserving *-automorphism of M, then α and β are *conjugate* if there exists a *-automorphism τ of M such that $\alpha \circ \tau = \tau \circ \beta$. By ergodicity of β, τ is necessarily m-preserving (i.e., $m \circ \tau$ is a β-invariant probability measure absolutely continuous with respect to m, hence $m \circ \tau = m$). See [3]. Finally, α gives rise to a unitary operator U_α, with the property that $U_\alpha A U_\alpha^{-1} = \alpha(A)$, for all $A \in M$. For example, if α comes from the transformation T, then $Uf = f \circ T$, $f \in \mathcal{H}$.

We consider two operator algebras; $\mathfrak{A}(\alpha)$, the Banach algebra generated by M together with the nonnegative powers of U_α, and $\mathfrak{B}(\alpha)$, the C^*-algebra generated by M and U_α. From the relation $U_\alpha M = M U_\alpha$, it follows that the set $\mathfrak{A}_0(\alpha)$ of finite sums $A_0 + A_1 U_\alpha + \cdots + A_n U_\alpha^n$ ($A_i \in M$, $n \geqq 0$) is a dense subalgebra of $\mathfrak{A}(\alpha)$, and the set $\mathfrak{B}_0(\alpha)$ of sums $A_m U_\alpha^m + \cdots + A_n U_\alpha^n$ ($A_i \in M$, $m \leqq n$) is a dense *-subalgebra of $\mathfrak{B}(\alpha)$. An application of free action gives:

LEMMA 1.1. *$\sum_m^n A_k U_\alpha^k = 0$ implies $A_k = 0$ for all k.*

Hence, one can define a linear mapping Φ of $\mathfrak{B}_0(\alpha)$ onto M by $\Phi(\sum A_k U_\alpha^k) = A_0$. Φ has the following properties:

$$
(1.2) \quad
\begin{aligned}
&(1) \quad \Phi \circ \Phi = \Phi, \ \Phi(I) = I. \\
&(2) \quad \Phi(AT) = A\Phi(T), \textit{ for } A \in M, \ T \in \mathfrak{B}_0(\alpha). \\
&(3) \quad \Phi(T^*) = \Phi(T)^*, \ T \in \mathfrak{B}_0(\alpha). \\
&(4) \quad \Phi(M U_\alpha^n) = 0, \ n \neq 0. \\
&(5) \quad 0 \leqq \Phi(T^*)\Phi(T) \leqq \Phi(T^*T), \ T \in \mathfrak{B}_0(\alpha).
\end{aligned}
$$

Properties (1.2) are in themselves not enough to guarantee that Φ is bounded, since $\mathfrak{B}_0(\alpha)$ is not closed. One needs the following lemma, another consequence of free action.

LEMMA 1.3. *For every projection $P\neq 0$ in M, there exists a state ρ of $\mathcal{B}(\alpha)$ such that $\rho(P)=1$ and $\rho(AU_\alpha^n)=0$ for all $A\in M$, $n\neq 0$.*

PROPOSITION 1.4. $\|\Phi(T)\|\leq\|T\|$, *for all* $T\in\mathcal{B}_0(\alpha)$.

So Φ may be extended, by continuity, to a linear mapping of $\mathcal{B}(\alpha)$ onto M. Properties (1.2) are valid for the extended map, which we denote by the same letter Φ.

A positive linear map Ψ of one C^*-algebra into another is *faithful* if $\Psi(T^*T)=0$ implies $T=0$, for all T. The proof that Φ is faithful is accomplished as follows. One constructs another C^*-algebra $\mathcal{C}$, a positive faithful linear map ω of $\mathcal{C}$ on M, and a *-representation π of $\mathcal{C}$ on $\mathcal{B}(\alpha)$, such that $\Phi\circ\pi=\omega$. This gives the result, for if $T\in\mathcal{B}(\alpha)$ and $\Phi(T^*T)=0$, then write $T=\pi(C)$, where C is some element of $\mathcal{C}$. Hence, $\omega(C^*C)=\Phi\circ\pi(C^*C)=\Phi(T^*T)=0$. Since ω is faithful, $C=0$, hence $T=\pi(C)=0$.

To sketch the construction, let Γ be the unit circle, and let $\mathcal{C}_1$ be the collection of all functions F defined on Γ, taking values in the set of bounded operators on $\mathcal{H}$, and which are norm-continuous on Γ. Endowed with the pointwise operations and the norm $\|F\| =\sup\|F(e^{i\theta})\|$, $\mathcal{C}_1$ becomes a C^*-algebra. Define $\pi(F)=F(1)$, and

$$\omega(F)=\frac{1}{2\pi}\int_0^{2\pi}F(e^{i\theta})d\theta.$$

Then π is a *-representation of $\mathcal{C}_1$ on the ring of all bounded linear operators on $\mathcal{H}$, and ω is a faithful positive linear map of $\mathcal{C}_1$ into the same thing. Now let $\mathcal{C}$ be the sub-C^*-algebra generated by the functions $F(e^{i\theta})=A_mU_\alpha^me^{im\theta}+\cdots+A_nU_\alpha^ne^{in\theta}$, where $A_k\in M$ and $m\leq n$. One shows that $\mathcal{C}$, and the restrictions of π and ω to $\mathcal{C}$ have the required properties. Thus we have:

THEOREM 1.5. *The extended map Φ is faithful on $\mathcal{B}(\alpha)$.*

We turn now to the algebra $\mathcal{A}(\alpha)$. If A, $B\in M$, and m and n are nonnegative integers, then $AU_\alpha^mBU_\alpha^n=A\alpha^m(B)U_\alpha^{m+n}$. Hence, $\Phi(AU_\alpha^mBU_\alpha^n)=0$ or AB according as $m+n>0$ or $m=n=0$. Thus $\Phi(AU_\alpha^mBU_\alpha^n)=\Phi(AU_\alpha^m)\Phi(BU_\alpha^n)$. It follows that Φ is a homomorphism of $\mathcal{A}(\alpha)$ onto M. If A is a self-adjoint operator in $\mathcal{A}(\alpha)$, and if we let $T=A-\Phi(A)$, then T is self-adjoint, belongs to $\mathcal{A}(\alpha)$, and $\Phi(T^2) =\Phi(T)^2=(\Phi(A)-\Phi\circ\Phi(A))^2=0$. By (1.5), $T=0$, and so $A=\Phi(A)\in M$.

As an immediate consequence, one has:

LEMMA 1.6. $\mathcal{A}(\alpha)\cap\mathcal{A}(\alpha)^*=M$.

The following lemma gives the decisive step in the proof of Theorem 1.8.

LEMMA 1.7. *Let V be a unitary operator in $\mathfrak{A}(\alpha)$ such that*

(1) *$\sigma: A \rightarrow VAV^{-1}$ is a freely-acting automorphism of M, and*

(2) *M and V generate $\mathfrak{A}(\alpha)$ as a Banach algebra.*

Then there exists a unitary W in M such that $V = U_\alpha W$. In particular, $\sigma = \alpha$.

THEOREM 1.8. *Let α and β be ergodic m-preserving $*$-automorphisms of M. Then α and β are conjugate if, and only if, there exists a unitary operator V such that $V\mathfrak{A}(\alpha)V^{-1} = \mathfrak{A}(\beta)$.*

REMARKS. Care must be exercised in closing the original algebra $\mathfrak{A}_0(\alpha)$. Indeed, $\mathfrak{A}_0(\alpha)$ contains the maximal abelian von Neumann algebra M, and it is easy to see that $\mathfrak{A}_0(\alpha)$ has no proper closed invariant subspaces. By the results of [2], $\mathfrak{A}_0(\alpha)$ is strongly dense in the ring of all bounded operators. In particular, the strong closure of $\mathfrak{A}_0(\alpha)$ contains no information about α.

Despite this, $\mathfrak{A}(\alpha)$ carries considerable structure. It is, on the one hand, a norm-closed irreducible triangular algebra in the sense of [5] (not necessarily maximal triangular), and on the other, it is closely related to the finite subdiagonal algebras studied in [1]. Note, for example, that $\mathfrak{A}(\alpha) + \mathfrak{A}(\alpha)^*$ is norm-dense in $\mathfrak{B}(\alpha)$.

2. **Generalizations.** We shall indicate to what extent the results of §1 generalize. Let M be the multiplication algebra of §1, and let G be an abstract group. By an *action* of G we mean a homomorphism $x \rightarrow \alpha_x$ of G into the group of $*$-automorphisms of M. We require that the α_x's preserve m, and that the action be *free* in the sense that for every $x \in G$, $x \neq e$, and every projection $P \neq 0$ in M, there exists a nonzero subprojection Q of P such that $\alpha_x(Q) \perp Q$. There is a naturally associated unitary representation U_x of G such that $U_x A U_x^{-1} = \alpha_x(A)$, for all A in M.

An application of free action gives the following: if $x_1, \cdots, x_n$ are distinct elements of G and $A_1, \cdots, A_n$ belong to M, then $A_1 U_{x_1} + \cdots + A_n U_{x_n} = 0$ implies $A_1 = \cdots = A_n = 0$. This allows one to define Φ on the $*$-algebra of finite sums $\sum A_{x_i} U_{x_i}$ exactly as in §1. Let $\mathfrak{B}$ be the norm closure of this algebra.

PROPOSITION 2.1. *Φ extends to a positive linear mapping of $\mathfrak{B}$ onto M, having the properties* (1.2) (1), (2), (3), *and* (5), *and* $\Phi(MU_x) = 0$, *for every $x \neq e$.*

One might expect an analog of Theorem 1.5 to be valid as well. This need not be so. Curiously, the criterion involves the structure of G itself, and does not depend on the properties of the particular action at hand.

THEOREM 2.2. *In order that the extension of Φ be faithful, it is necessary and sufficient that G admit an invariant mean.*

While space does not permit us to outline the argument for Theorem 2.2, it is perhaps worth noting that the proof makes no use of an invariant functional itself, but rather the structural conditions imposed on G by the existence of one (see [**4**] and [**6**]).

REFERENCES

1. W. B. Arveson, *Analyticity in operator alegbras*, (to appear).

2. ———, *A density theorem for operator algebras*, (to appear).

3. H. A. Dye, *On groups of measure preserving transformations*, Amer. J. Math. **81** (1959), 119–159.

4. E. Følner, *On groups with full Banach mean values*, Math. Scand. **3** (1955), 243–254.

5. R. Kadison and I. Singer, *Triangular operator algebras*, Amer. J. Math. **82** (1960), 227–259.

6. I. Namioka, *Følner's condition for amenable semi-groups*, Math. Scand. **15** (1964), 18–28.

HARVARD UNIVERSITY