ON LATTICE-POINTS IN A RANDOM SPHERE

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Communicated by A. A. Albert, September 15, 1966

1. I. M. Vinogradov and A. G. Postnikov, in their one-hour report on *Recent developments in analytic number theory* at the International Congress of Mathematicians (Moscow, 1966) have referred to a recent result of A. A. Judin on the lattice-point problem for a random circle. If (α, β) is an arbitrary point in the plane, and $A(x; \alpha, \beta)$ denotes the number of lattice-points inside and on the circumference of a circle with (α, β) as centre and $x^{1/2}$ as radius, then Judin's result, as stated in the above-mentioned report, is that

$$\limsup_{x \to \infty} \frac{|A(x; \alpha, \beta) - \pi x|}{x^{1/4}} > c > 0,$$

and the proof, according to the report, is by the application of arguments from the theory of almost periodic functions. This is of interest in view of the known result [3] that

$$A(x; \alpha, \beta) - \pi x = O(x^{1/4+\epsilon}), \quad \epsilon > 0,$$

for almost all points (α, β).

It is our object to show that the following result, hence also Judin's, is a direct consequence of a theorem of ours on the average order of arithmetical functions:

$$\limsup_{x \to \infty} \frac{A(x; \alpha, \beta) - \pi x}{x^{1/4}} > 0,$$

$$\liminf_{x \to \infty} \frac{A(x; \alpha, \beta) - \pi x}{x^{1/4}} < 0.$$

This result is true not only in the plane, but in k dimensions, for $k \geqq 2$. Instead of $A(x; \alpha, \beta)$, one can also consider its higher averages of order $\rho \geqq 0$, the proof being the same.

2. THEOREM. *If $(\alpha_1, \cdots, \alpha_k)$ is an arbitrary point in k-space, $k \geqq 2$, and $A(x; \alpha_1, \cdots, \alpha_k)$ denotes the number of lattice-points inside and on a sphere with centre $(\alpha_1, \cdots, \alpha_k)$, and radius $x^{1/2}$, then*

$$[A(x; \alpha_1, \cdots, \alpha_k) - \pi^{k/2}x^{k/2}/\Gamma(k/2 + 1)] = \Omega_{\pm}(x^{(k-1)/4}),$$

as $x \to \infty$.

PROOF. (i) Let $\alpha_1, \cdots, \alpha_k$ be given real numbers, not all being integers at the same time. Let (n_k) denote integers. Let (λ_r) be the sequence of real numbers $\{(n_1-\alpha_1)^2+\cdots+(n_k-\alpha_k)^2\}$ arranged in increasing order of magnitude. Define

$$a_r = \sum_{(n_1-\alpha_1)^2+\cdots+(n_k-\alpha_k)^2=\lambda_r} 1.$$

Consider the Dirichlet series

$$\begin{aligned}\phi(s) &= \sum_{r=1}^{\infty} \frac{a_r}{\lambda_r^s}, \qquad s = \sigma + it,\\ &= \sum_{n_r=-\infty}^{\infty} \cdots \sum \frac{1}{\{(n_1-\alpha_1)^2+\cdots+(n_k-\alpha_k)^2\}^s}.\end{aligned}$$

This converges absolutely for $\sigma > k/2$, and satisfies a functional equation given by

$$\pi^{-s}\Gamma(s)\phi(s) = \pi^{s-k/2}\Gamma\left(\frac{k}{2}-s\right)\psi\left(\frac{k}{2}-s\right), \tag{2.1}$$

where ψ is represented by the Dirichlet series

$$\begin{aligned}\psi(s) &= \sum_{r=1}^{\infty} \frac{b_r}{r^s}\\ &= \sum_{n_r=-\infty;\ (n_1,\cdots,n_k)\neq(0,0,\cdots,0)}^{\infty} \cdots \sum \frac{\exp\,(2\pi i(n_1\alpha_1+\cdots+n_k\alpha_k))}{(n_1^2+\cdots+n_k^2)^s},\end{aligned}$$

where

$$b_r = \sum_{n_1^2+\cdots+n_k^2=r} \exp\,(2\pi i(n_1\alpha_1+\cdots+n_k\alpha_k)).$$

Equation (2.1) can be proved directly in the same way as the functional equation of Riemann's zeta-function. If

$$\theta(\alpha, y) = \sum_{n_r=-\infty}^{\infty} \cdots \sum \exp\,(-[(n_1-\alpha_1)^2+\cdots+(n_k-\alpha_k)^2]\pi y),$$

for $\operatorname{Re} y > 0$, then

$$\theta(\alpha, y) = y^{-k/2}\theta_1(\alpha, 1/y),$$

where

$$\theta_1(\alpha, y)$$
$$= \sum_{n_r=-\infty}^{\infty} \cdots \sum \exp\,(2\pi i(n_1\alpha_1+\cdots+n_k\alpha_k) - \pi(n_1^2+\cdots+n_k^2)y).$$

If we denote

$$\theta_2(\alpha, y) = \theta_1(\alpha, y) - 1,$$

then, for $\sigma > k/2$, we have

$$\begin{aligned}\pi^{-s}\Gamma(s)\phi(s) &= \int_1^\infty y^{s-1}\theta(\alpha, y)dy \\ &\quad + \int_1^\infty y^{k/2-s-1}\theta_2(\alpha, y)dy + \frac{1}{s-\frac{1}{2}k},\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\pi^{-s}\Gamma(s)\psi(s) &= \int_1^\infty y^{s-1}\theta_2(\alpha, y)dy \\ &\quad + \int_1^\infty y^{k/2-s-1}\theta(\alpha, y)dy - \frac{1}{s}.\end{aligned} \tag{2.3}$$

These two relations show that $\phi(s)$ and $\psi(s)$ are meromorphic functions in the whole s-plane, with ϕ having a simple pole at $s=k/2$ with residue $\pi^{k/2}/\Gamma(k/2)$. Further ϕ and ψ satisfy equation (2.1). Not all the coefficients (b_n) are zero. Hence Theorem 3.2 of [2] is applicable, with $\rho=0$, $Q_0(x)=\pi^{k/2}x^{k/2}/\Gamma(k/2+1)$, and $\theta=(k-1)/4$, giving what we want.

(ii) If $\alpha_1, \cdots, \alpha_k$ are all integers, then $\phi(s)=\psi(s)$, and we have Epstein's zeta-function, which is known to satisfy (2.1). The result is again obvious.

REMARK 1. If one starts with a positive-definite quadratic form Q in k-variables, with real coefficients, where $k \geqq 2$, one considers the corresponding function

$$A(x; Q, \alpha) = \sum_{Q(n-\alpha)\leqq x} 1,$$

and obtains the result

$$A(x; Q, \alpha) - \pi^{k/2}x^{k/2}/\Gamma(k/2+1)\,|\,Q\,|^{1/2} = \Omega_\pm(x^{(k-1)/4}),$$

as $x \to \infty$, where $|Q|$ is the determinant of Q.

REMARK 2. The function $A(x; \alpha_1, \cdots, \alpha_k)$ is integrable and multiperiodic in the α's, with period 1, and its Fourier expansion is given by

$$A(x; \alpha_1, \cdots, \alpha_k) \sim c_1x^{k/2} + c_2x^{k/4}\sum \cdots \sum$$

$$\cdot \frac{\exp(2\pi i(\alpha_1 n_1 + \cdots + \alpha_k n_k))J_{k/2}(2\pi x^{1/2}(n_1^2 + \cdots + n_k^2)^{1/2})}{(n_1^2 + \cdots + n_k^2)^{k/4}}.$$

If one integrates the series on the right, with respect to x, ρ times, where ρ is a sufficiently large integer, one obtains an absolutely convergent series, which is the Fourier series of $A_\rho(x;\alpha_1,\cdots,\alpha_k)$, the ρth integral, with respect to x, of $A(x;\alpha_1,\cdots,\alpha_k)$, and is therefore equal to it. Thus one obtains an identity of the form

$$\frac{1}{\Gamma(\rho)}\int_0^x A(t;\alpha_1,\cdots,\alpha_k)(x-t)^{\rho-1}dt$$

$$= c_3x^{k/2+\rho} + c_4x^{k/4+\rho/2}\sum\cdots\sum\frac{J_{k/2+\rho}(2\pi x^{1/2}(n_1^2+\cdots+n_k^2)^{1/2})}{(n_1^2+\cdots+n_k^2)^{k/4+\rho/2}}$$

$$\cdot\exp\left(2\pi i(\alpha_1n_1+\cdots+\alpha_kn_k)\right).$$

It is known that this is equivalent to a functional equation of the form (2.1). (See Lemma 5 of [1].)

References

1. K. Chandrasekharan and Raghavan Narasimhan, *Hecke's functional equation and arithmetical identities*, Ann. of Math. **74** (1961), 1–23.

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