ON THE HAUPTVERMUTUNG, TRIANGULATION OF MANIFOLDS, AND h-COBORDISM

BY R. LASHOF AND M. ROTHENBERG

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We consider the question of uniqueness and existence of piecewise linear structures on manifolds.

I. **Some relations between existence and uniqueness.** By a manifold we will, in general, mean a topological manifold with or without boundary, compact or not. A PL manifold will be a topological manifold along with a given triangulation as a combinatorial manifold. A PL map will be the usual thing. If M is a manifold, $t(M)$ will denote its topological tangent bundle [**1**]. A tangential equivalence $f: M \to M'$ will be a homotopy equivalence such that $t(M)$ and $f^*t(M')$ are stably equivalent. An h-cobordism, W, will be a compact manifold with $\partial W = \partial_0 W \cup \partial_1 W$, where $\partial_i W$ are the components of ∂W such that there exists a manifold M and a homotopy equivalence

$$f: (W, \partial_0 W \cup \partial_1 W) \to (M \times I, M \times (0) \cup M \times (1)).$$

$\partial_0 W$ and $\partial_1 W$ are said to be h-cobordant. $[X, Y]$ will denote the set of homotopy classes of maps.

DEFINITION. *The closed manifold M satisfies condition α_n^k if*

(a) $\dim M \geqq k$

(b) *M is n-connected if $n > 0$, $\pi_1(M)$ is free abelian and finitely generated if $n = 0$.*

Consider the following statements:

A_n^k—Every closed manifold satisfying α_n^k is homeomorphic to a PL manifold.

B_n^k—If M^1, M^2 satisfy α_n^k and M^1, M^2 are h-cobordant then M^1 is homeomorphic to M^2.

C_n—For each n-connected closed manifold M, there exists an l such that $M \times R^l$ is homeomorphic to a PL manifold.

THEOREM A. $A_n^k \Leftrightarrow B_n^k + C_n$.

Now consider the statement:

D_n^k—If M^1, M^2 are simply connected PL manifolds satisfying α_n^k and if M^1, M^2 are h-cobordant as topological manifolds, then there exists a PL isomorphism between M^1, M^2.

THEOREM B. $C_n \Rightarrow D_n^k$.

Roughly speaking, Theorem B says that if every closed manifold has a stable PL-structure, then any PL-structure on a closed manifold is unique.

Theorem A shows the relation between PL-structures and stable PL-structures.

In our second example of an existence implies uniqueness theorem, we weaken both hypothesis and conclusion:

E—For every closed manifold M, $\pi_1(M)=Z$, there is a closed PL-manifold N with $M\times N$ of the same homotopy type as a closed PL manifold.

F—If M^1, M^2 are simply connected tangentially h-cobordant PL-manifolds of dimension $\geqq 5$, then M^1 is PL isomorphic to M^2. (Tangentially h-cobordant means there is a topological h-cobordism W and a PL-bundle $\mathcal{E}$ over W with $\mathcal{E}|M^i$, $i=1, 2$, stably equivalent to the tangent bundle of M^i.)

THEOREM C. $E\Rightarrow F$.

Roughly, existence up to homotopy implies homeomorphic PL-manifolds with stably equivalent tangent bundles are PL-isomorphic.

REMARK. Browder and Hirsch have independently obtained a result similar to Theorem B (unpublished).

II. **Homotopy theoretic interpretation.** We assume familiarity with semisimplicial groups PL, TOP. Let G_n be the singular complex of the space of homotopy equivalences of S^n leaving basepoint fixed. G_n is an associative, complete, semisimplicial monoid and $G_n\subset G_{n+1}$ by suspension. Let $G=\bigcup_{n=0}^{\infty} G_n$. Then we have $\mathrm{PL}\subset\mathrm{TOP}\subset G$. Recall that $\pi_r(G)=\pi_{r+k}(S^k)$, $k\gg r$.

THEOREM D. *Let M be a closed, simply connected* PL *manifold with* dim $M\geqq 5$. *Then*

(a) *If* $[M, G/\mathrm{PL}]\rightarrow[M, G/\mathrm{TOP}]$ *is injective, any closed* PL *manifold M', topologically h-cobordant to M, is* PL *isomorphic to M.*

(b) *If the composite* $[M, \mathrm{TOP}]\rightarrow[M, \mathrm{TOP}/PL]\rightarrow[M, G/\mathrm{PL}]$ *is* 0, *then any* PL *manifold M' tangentially h-cobordant to M is* PL *isomorphic to M.*

REMARK. (a) follows from (b).

If W is any of the complexes PL, TOP, G, G/PL, G/TOP, TOP/PL, BTOP, BG, BPL, then the functor $[\ ,W]$ has the natural structure of an abelian group. Theorem D is an easy consequence of the main theorem of **[3]**; however, we have given a somewhat different direct proof.

III. **A refined splitting theorem and some applications.** The following is a slight refinement of the splitting theorem. Its proof is very similar, see [**2**].

THEOREM E. *Let M be a closed topological manifold of dimension $\geqq 5$ and such that $\pi_1(M)$ is finitely generated and free abelian. Suppose that W is a* PL *manifold and $h: W \to M \times R^k$ is a homeomorphism. Then there exists a* PL *manifold M' and maps $\lambda: M' \to M$, $d: M' \times R^k \to W$ such that*

1. *d is a* PL *isomorphism;*
2. *the map $\lambda \times id: M' \times R^k \to M \times R^k$ is homotopic, through proper maps, to hd.*

This leads to:

THEOREM F. *Let M^n be a closed* PL *orientable manifold, $n \neq 2$. Let $M^n \xrightarrow{\gamma} S^n$ be the map of degree* 1. *Let $f: S^n \to G/\mathrm{PL}$ and consider the composite $\mu: M^n \to G/\mathrm{TOP}$ defined by*

$$M^n \xrightarrow{\gamma} S^n \xrightarrow{f} G/\mathrm{PL} \to G/\mathrm{TOP}.$$

If μ is homotopic to a constant then so is f.

COROLLARY 1. *For $n \neq 2$ one has $0 \to \pi_n(G/\mathrm{PL}) \to \pi_n(G/\mathrm{TOP})$.*

REMARK. We have not been able to settle the case $n=2$. Theorem F follows fairly directly from Theorem E and a knowledge of the groups $\pi_n(G/\mathrm{PL})$. These groups have been computed in [**3**].

COROLLARY 2. *Let M^n be a closed* PL *orientable manifold, $n \neq 2$, and let $p \in M^n$. If $[M^n - p, G/\mathrm{PL}] \to [M^n - p, G/\mathrm{TOP}]$ is injective, then $[M^n, G/PL] \to [M^n, G/\mathrm{TOP}]$ is injective.*

An easy explicit computation shows that $[CP_2, G/\mathrm{PL}] \to [CP_2, G/\mathrm{TOP}]$ is injective.

One can show, using the above computation and Corollary 2 that:

THEOREM G. *If $M = \times_i M_i$, where each $M_i = S^n$, $n > 2$ or $M = CP_k$, $k > 1$, or $M = QP_k$ (quaternionic projective space), then $[M, G/\mathrm{PL}] \to [M, G/\mathrm{TOP}]$ is injective.*

COROLLARY TO G. *Combining* G *with Theorem* D *part* (a), *one sees that if M is as in Theorem* G *and M' is topologically h-cobordant to M then M' is* PL *isomorphic to M.*

IV. **An example.** Sullivan has constructed a differentiable manifold M^8 tangentially equivalent to CP_4, but not PL isomorphic to

CP_4. By the Corollary to G, M^8 is not topologically h-cobordant to CP_4 hence is not homeomorphic to CP_4. However, from tangential equivalence and simple connectivity follows the fact that $CP_4 \times D^k$ is diffeomorphic to $M^8 \times D^k$. Hence CP_4, M^8 are simply connected manifolds which are not homeomorphic; yet $CP_4 \times D^k$ and $M^8 \times D^k$ are diffeomorphic. This is the first such example known.

References

1. J. Milnor, *Microbundles*. I, Topology **3** (1964), 53–80.
2. J. Milnor, et al., *Pontryagin classes for topological manifolds*, (to appear).
3. D. Sullivan, Ph.D. Thesis, Princeton University, 1966.

University of Chicago