IDEALS WITH SMALL AUTOMORPHISMS

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In [1], Forelli proves the following: If G_1 and G_2 are locally compact Abelian groups, if J is a closed ideal in the group algebra $L^1(G_1)$, and if Ψ is a homomorphism of J into the measure algebra $M(G_2)$ with $\|\Psi\|=1$, then Ψ is induced by an affine map of a coset in Γ_2 into Γ_1. (See [1] for a more detailed statement. For notation and terminology, see [1] or [2]; Γ_i denotes the dual group of G_i; the circle group will be denoted by T.) As Forelli points out in [1], the assumption $\|\Psi\|=1$ cannot be entirely discarded.

Actually, the assumption $\|\Psi\|=1$ cannot even be replaced by $\|\Psi\| <1+\epsilon$, no matter how small $\epsilon>0$ is, even if "affine" is replaced by "piecewise affine" in the conclusion, and even if $G_1=G_2=T$ and Ψ is assumed to be one-to-one.

Since the integer group Z admits only countably many piecewise affine maps, the preceding statement is a consequence of the theorem below. By way of contrast, it may be of interest to mention that if Ψ is a homomorphism of all of $L^1(G_1)$ into $M(G_2)$ and if $\|\Psi\|>1$, then $\|\Psi\|\geqq\sqrt{5}/2$ [2, p. 88].

THEOREM. *Suppose $0<\epsilon<1$. Let E be a set of positive integers λ_k such that $\lambda_1=1$ and*

$$\sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_{k+1}}<\frac{\epsilon}{6\pi}\,. \tag{1}$$

Let J be the set of all $f\in L^1(T)$ whose nth Fourier coefficient $\hat{f}(n)$ is 0 for all n not in E. Then J is a closed ideal in $L^1(T)$, with continuum many automorphisms, and every automorphism A of J (other than the identity) satisfies the inequality

$$1<\|A\|<1+\epsilon. \tag{2}$$

We shall sketch the proof.

Each A is induced by a permutation α of E. The gaps in E show that no affine map (other than the identity) carries E onto E. Thus $\|A\|>1$ if $A\neq I$.

We write $e(t)$ in place of $e^{2\pi i t}$.

Let α be any permutation of Z^+ (the positive integers), let

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$$f(t) = \sum c(k)e(\lambda_k t), \qquad g(t) = \sum c(\alpha(k))e(\lambda_k t) \tag{3}$$

be trigonometric polynomials in J. The theorem is an immediate consequence of the inequality

$$\|g\|_1 \leqq \left(1 + \frac{7\epsilon}{9}\right)\|f\|_1. \tag{4}$$

To prove (4), fix N so that $\hat{f}$ and $\hat{g}$ have their supports in $\{\lambda_1, \cdots, \lambda_N\}$. For $0 \leqq t < 1$, let D_t be the set of all $x = (x_1, \cdots, x_N)$ in R^N such that

$$\begin{cases} \lambda_k t \leqq x_k < \lambda_k t + \lambda_k/\lambda_{k+1} & (1 \leqq k \leqq N-1), \\ \lambda_N t = x_N, \end{cases} \tag{5}$$

and let Q be the union of these $(N-1)$-cells D_t. We claim that Q *contains no point of* Z^N, *except* 0: Assume, to get a contradiction, that $x \in Q \cap Z^N$, $x \neq 0$.

If $x_1 = 0$, (5) implies $t = 0$, hence $x_2 = \cdots = x_N = 0$. So $x_1 = 1 = \lambda_1$. If $2 \leqq k \leqq N$ and $x_{k-1} = \lambda_{k-1}$, (5) gives

$$\lambda_{k-1} < \lambda_{k-1}t + \lambda_{k-1}/\lambda_k \leqq \lambda_{k-1}(1 + x_k)/\lambda_k,$$

or $\lambda_k < 1 + x_k$. Since $x_k \leqq \lambda_k$, we have $x_k = \lambda_k$. This leads to $x_N = \lambda_N$, a contradiction to the last equation (5).

Since Q is a parallelopiped with one vertex at 0 it now follows that no two points of Q are congruent modulo Z^N. Also, Q has volume 1. Thus if we regard functions on the torus T^N as periodic functions on R^N, with period 1 in each of the variables $x_1, \cdots, x_N$, *integration over* T^N *may be replaced by integration over* Q.

We return to our polynomials (3) and define

$$F(x) = \sum_1^N c(k)e(x_k), \qquad G(x) = \sum_1^N c(\alpha(k))e(x_k) \quad (x \in R^N). \tag{6}$$

These are trigonometric polynomials on T^N. Clearly

$$\|F\|_1 = \|G\|_1. \tag{7}$$

For $x \in D_t$, put $\tilde{F}(x) = f(t)$, $\tilde{G}(x) = g(t)$. This defines $\tilde{F}$ on Q, hence on T^N; (5) and (1) imply that

$$\begin{aligned} |\tilde{F}(x) - F(x)| &\leqq \sum_1^N |c(k)|\,|e(\lambda_k t) - e(x_k)| \\ &\leqq 2\pi \sum_1^N |c(k)|\,\lambda_k/\lambda_{k+1} \leqq \frac{\epsilon}{3}\|f\|_1 \end{aligned} \tag{8}$$

if $x \in D_t$. Since $\|\tilde{F}\|_1$ can be computed by integrating $|\tilde{F}|$ over Q, the definition of $\tilde{F}$ shows, via Fubini's theorem, that $\|\tilde{F}\|_1 = \|f\|_1$. By (8) this gives

$$\|F\|_1 \leqq \left(1 + \frac{\epsilon}{3}\right) \|f\|_1. \tag{9}$$

The inequality $\|g\|_1 \leqq (1+\epsilon/3)\|G\|_1$ is obtained in the same way; combined with (7) and (9) it yields (4).

REMARK. If $E = \{\lambda_k\}$ is as in the theorem, if $1 \leqq p \leqq \infty$, if $\sum c(k)e(\lambda_k t)$ is the Fourier series of some $f \in L^p(T)$, and if α is any permutation of Z^+, the above proof also shows that $\sum c(\alpha(k))e(\lambda_k t)$ is the Fourier series of a function $g \in L^p(T)$, and that $\|g\|_p \leqq (1+\epsilon)\|f\|_p$.

REFERENCES

1. Frank Forelli, *Homomorphisms of ideals in group algebras*, Illinois J. Math. 9 (1965), 410–417.

2. Walter Rudin, *Fourier analysis on groups*, Interscience, New York, 1962.

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