ON THE SPECTRUM OF GENERAL SECOND ORDER OPERATORS[1]

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Let λ_1 be the lowest eigenvalue of the membrane problem

$$\Delta u + \lambda u = 0 \quad \text{in } D,$$
$$u = 0 \quad \text{on } \partial D.$$

It was shown by Barta [1] that if $w>0$ in D, then

$$\lambda_1 \geqq \inf \left[-\frac{\Delta w}{w} \right].$$

This result has been extended to other selfadjoint problems for second order operators. See [2], [3], and [6].

The purpose of this note is to show that the same technique locates the spectrum of a nonselfadjoint problem in a half-plane. Such a result is of interest in investigating stability, where one needs to know whether there is any spectrum in the half-plane $\operatorname{Re} \lambda \leqq 0$.

In a bounded domain D we consider the differential equation

$$\tag{1} \begin{aligned} L[u] + \lambda k u \equiv \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x) u + \lambda k(x) u \\ = - k(x) f(x) \end{aligned}$$

where $x \sim (x_1, \cdots, x_n)$. The matrix $a^{ij}(x)$ is symmetric and positive definite, $k(x)$ is positive, and all the coefficients are real and bounded in D. However, they need not be continuous.

The boundary ∂D is divided into two disjoint parts Σ_1 and Σ_2, and the boundary conditions are

$$\tag{2} \begin{aligned} u = 0 \quad \text{on } \Sigma_1, \\ M[u] \equiv \sum_{1}^{n} e^i(x) \frac{\partial u}{\partial x_i} + g(x) u = 0 \quad \text{on } \Sigma_2. \end{aligned}$$

The vector field $\boldsymbol{e}$ points outward from D.

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We shall prove the following theorem about the spectrum of the operator L considered as an operator on the space $C(D)$ of continuous functions with the maximum norm.

THEOREM 1. *Suppose $w(x)$ defined on $D\cup\partial D$ has the properties:*
(i) $w(x)>0$ *on* $D\cup\partial D$;
(ii) $w\in C^2(D)\cap C^1(D\cup\partial D)$;
(iii) $M[w]\geqq 0$ *on* Σ_2.
Then the discrete and continuous spectra of the problem (1), (2) *are contained in the half-plane*

$$\operatorname{Re}\lambda \geqq \inf\left(-\frac{L[w]}{kw}\right). \tag{3}$$

PROOF. Let $\tau=\inf(-L[w]/kw)$, and suppose that $\operatorname{Re}\lambda<\tau$. We wish to show that λ is in the resolvent set.

Let u satisfy (1) and (2), and define

$$v(x) \equiv u(x)/w(x).$$

Substituting $u=vw$ in (1), multiplying the equation by $\bar v$, and taking real parts, we obtain

$$\begin{aligned}\sum_1^n \frac{1}{2}wa^{ij}\frac{\partial^2|v|^2}{\partial x_i\partial x_j}+\sum_{i=1}^n\frac{1}{2}\left(wb^i+2\sum_{j=1}^n a^{ij}\frac{\partial w}{\partial x_j}\right)\frac{\partial|v|^2}{\partial x_i}&\\ +(L[w]+\operatorname{Re}(\lambda)kw)|v|^2&\\ =\sum_1^n a^{ij}\frac{\partial v}{\partial x_i}\frac{\partial\bar v}{\partial x_j}-\operatorname{Re}(\bar v f)&\\ \geqq -k\operatorname{Re}(\bar v f),&\end{aligned}$$

since a^{ij} is positive definite. The boundary conditions yield

$$|v|^2=0 \quad\text{on } \Sigma_1,$$
$$\sum_1^n e^i\frac{\partial|v|^2}{\partial x_i}+2M[w]|v|^2=0\quad\text{on } \Sigma_2.$$

We observe that $L[w]+\operatorname{Re}(\lambda)kw\leqq-(\tau-\operatorname{Re}\lambda)kw$. Therefore by the maximum principle, we find that

$$|v|^2\leqq\frac{1}{\tau-\operatorname{Re}\lambda}\sup_D\frac{\operatorname{Re}(\bar v f)}{w}.$$

Hence

$$|v| \leqq \frac{1}{\tau - \operatorname{Re}(\lambda)} \sup_D \frac{|f|}{w},$$

and

$$|u| \leqq \frac{w}{\tau - \operatorname{Re}(\lambda)} \sup_D \frac{|f|}{w}.$$

Thus if $\operatorname{Re}\lambda < \tau$, the operator $L+\lambda k$ has a bounded inverse in the maximum norm on its range. Hence λ is in either the residual spectrum or the resolvent set. Therefore the discrete and continuous spectra are contained in the half-plane

$$\operatorname{Re}\lambda \geqq \inf_D \left(-\frac{L[w]}{kw} \right)$$

as the theorem states.

In what follows we shall assume that the problem does not have a residual spectrum. That is, we assume that the range of $L+\lambda k$ is dense for some sufficiently small λ; or, equivalently, that the index is zero.

The following theorem shows that the bound (3) is a lower bound for a real point λ_1 of the spectrum:

THEOREM 2. *Suppose there is a function w satisfying the conditions of Theorem* 1. *Then if the spectrum of* (1), (2) *is not empty, there exists a real number λ_1 in the spectrum such that the whole spectrum lies in the half-plane*

$$\operatorname{Re}\lambda \geqq \lambda_1.$$

PROOF. Let λ be real, and let $v = u/w$, where u is real. Then the problem (1), (2) becomes

$$\sum_1^n wa^{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n \left(wb^i + 2 \sum_{j=1}^n a^{ij} \frac{\partial w}{\partial x_j} \right) \frac{\partial v}{\partial x_i} + (L[w] + \lambda kw)v = -kf \quad \text{in } D,$$

$$v = 0 \quad \text{on } \Sigma_1,$$

$$\sum_1^n we^i \frac{\partial v}{\partial x_i} + M[w]v = 0 \quad \text{on } \Sigma_2.$$

By the maximum principle we see that if $\lambda < \inf(-L[w]/kw)$, then $f > 0$ implies $v > 0$ and hence $u > 0$. Thus the resolvent R_λ is positive for $\lambda < \inf(-L[w]/kw)$.

Conversely if $R_\tau \geqq 0$ for some real number τ, we find that the solution w of

$$
\begin{aligned}
L[w] + \tau k w &= -k && \text{in } D, \\
w &= 1 && \text{on } \Sigma_1, \\
M[w] &= 0 && \text{on } \Sigma_2,
\end{aligned}
\tag{4}
$$

is admissible in Theorem 1, so that the spectrum lies in the half-plane $\operatorname{Re}\lambda \geqq \tau$, and $R_\mu \geqq 0$ for all real $\mu \leqq \tau$.

Now let λ_1 be the limit superior of those λ for which $R_\lambda \geqq 0$. Then the spectrum is in the half-plane $\operatorname{Re}\lambda \geqq \lambda_1$. If λ_1 is in the resolvent set, we see by continuity that $R_{\lambda_1} \geqq 0$. Moreover, for any $\lambda > \lambda_1$ with $\lambda - \lambda_1 < \|R_{\lambda_1}\|^{-1}$ we have $R_\lambda = (I - (\lambda - \lambda_1)R_{\lambda_1})^{-1} R_{\lambda_1} = R_{\lambda_1} + (\lambda - \lambda_1) R_{\lambda_1}^2 + \cdots \geqq 0$. Thus if λ_1 is in the resolvent set, we obtain a contradiction with the definition of λ_1. Hence λ_1 is in the spectrum of (1), (2).

We observe that for any $\tau < \lambda_1$ the solution w of (4) gives the lower bound τ, so that the lower bound (3) can be made arbitrarily close to λ_1 by a judicious choice of w.

REMARKS 1. If D is unbounded but Σ_2 is bounded, we can define a solution of (1), (2) by exhaustion. That is, we obtain the solutions u_n of

$$
\begin{aligned}
L[u_n] + \lambda k u_n &= -kf && \text{in } D \cap \{ |x| < n \}, \\
u_n &= 0 && \text{on } \Sigma_1 \cup \{ |x| = n \}, \\
M[u_n] &= 0 && \text{on } \Sigma_2.
\end{aligned}
$$

By the method used in the proof of Theorem 1 we find that if $\operatorname{Re}\lambda < \inf(-L[w]/kw)$, the functions u_n converge uniformly to a solution u of (1), (2). Thus the spectrum still lies in $\operatorname{Re}\lambda \geqq \inf(-L[w]/kw)$. Theorem 2 can also be extended to this case.

2. If D and the coefficients of our problem are so smooth that for sufficiently small real μ the resolvent R_μ is completely continuous in the maximum norm (i.e., the family $R_\mu[f]$ with $f \leqq 1$ is equicontinuous), then the spectrum is discrete, so that λ_1 is an eigenvalue.

A theorem of Kreĭn and Rutman [5, Theorem 6.1] shows that the corresponding eigenfunction u_1 is positive in D. The theorem of Kreĭn and Rutman also states that in this case the adjoint operator R_μ^* has the eigenvalue $(\lambda_1 - \mu)^{-1}$ with a positive eigenfunctional u_1^*. From this fact we can derive Theorem 1 with condition (i) replaced by the weaker condition $w \geqq 0$. Moreover, we can obtain a complementary upper bound for λ_1:

If $q(x) \geqq 0$ in D, $q = 0$ on Σ_1, and $M[q] \leqq 0$ on Σ_2, then $\lambda_1 \leqq \sup(-L[q]/kq)$.

3. If the coefficients are so smooth that the adjoint operator L^* can be formed, and if the boundary conditions are selfadjoint (e.g., $\Sigma_1 = \partial D$), an inequality of the same type as (3) may be found by methods of Hooker [**4**] and Protter [**6**]. Namely,

$$\mathrm{Re}(\lambda) \geqq \inf_D \left(- \frac{L[w] + L^*[w]}{2kw} \right).$$

This inequality may be stronger or weaker than (3).

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