A JORDAN DECOMPOSITION FOR OPERATORS IN BANACH SPACE

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Operators T with real spectrum in finite dimensional complex Euclidian space may be characterized by the property

$$|e^{itT}| = O(|t|^k), \qquad t \text{ real.} \tag{1}$$

Our result is a Jordan decomposition theorem for operators T in reflexive Banach space which satisfy (1) and whose spectrum (which is real because of (1)) has linear Lebesgue measure zero.

1. **The Jordan manifold.** Let X be a complex Banach space; denote by $B(X)$ the Banach algebra of all bounded linear operators acting on X. For $m=0, 1, 2, \cdots$, C^m is the topological algebra of all complex valued functions on the real line R with continuous derivatives up to the order m, with pointwise operations and with the topology of uniform convergence on every compact set of all such derivatives. Fix $T\in B(X)$. Following [3], we say that T is of class C^m if there exists a C^m-operational calculus for T, i.e., a continuous representation $f\to T(f)$ of C^m into $B(X)$ such that $T(1)=I$, $T(f)=T$ if $f(t)\equiv t$, and $T(\cdot)$ has compact support. The latter is then equal to the spectrum of T, $\sigma(T)$. It is known that if T satisfies (1), then it is of class C^m for $m\geqq k+2$ and has real spectrum (cf. Lemma 2.11 in [3]).

From now on, let $T\in B(X)$ satisfy (1), and let $T(\cdot)$ be the (unique) C^m-operational calculus for T, for m fixed $\geqq k+2$. We write:

1. $|f|_{m,T}=\sum_{j\leq m} \max_{\sigma(T)} |f^{(j)}|/j!$, $f\in C^m$;
2. $|x|_{m,T}=\sup\{|T(f)x|;f\in C^m, |f|_{m,T}\leqq 1\}$, $x\in X$;
3. $D_m=\{x\in X; |x|_{m,T}<\infty\}$;
4. $D=\bigcup_{m\geq k+2} D_m$.

We call *D the Jordan manifold for T.* It is an invariant linear manifold for any $V\in B(X)$ which commutes with T. If $\sigma(T)$ is a finite union of points and closed intervals, then there exists an $m\geqq k+2$ such that $D=D_m=X$. This is true for $m=k+2$ if $\sigma(T)$ is a finite point set. It follows in particular that D_{k+2} contains every finite dimensional invariant subspace for T, hence all the eigenvectors of T. It is also true that D contains all the root vectors for T, and is therefore dense in X if the root vectors are fundamental in X.

THEOREM 1. *Suppose that all nonzero points of $\sigma(T)$ are isolated.*

Then the closure of D_{k+2} contains the closed range of T^{k+1}. For $k=0$ and X reflexive, D_2 is dense in X.

2. **The Jordan decomposition.** If W is a linear manifold in X, we denote by $T(W)$ the algebra of all linear transformations of X with domain W and range contained in W.

Let B denote the Borel field of R.

A *generalized spectral measure on* W is a map $E(\cdot)$ of B into $T(W)$ such that

(i) $E(R)x=x$ for all $x\in W$, and

(ii) $E(\cdot)x$ is a bounded regular strongly countably additive vector measure on B, for each $x\in W$.

We can state now our generalization of the classical Jordan decomposition theorem for complex matrices with real spectrum to infinite dimensional Banach spaces.

THEOREM 2. *Let X be a reflexive Banach space. Let $T\in B(X)$ satisfy* (1). *Suppose $\sigma(T)$ (which lies on R because of* (1)) *has linear Lebesgue measure zero. Let D be the Jordan manifold for T. Then there exist S and N in $T(D)$ such that*

(a) $T/D=S+N$;

(b) $SN=NS$;

(c) $N^{k+1}=0$; *and*

(d) $p(S)x=\int_{\sigma(T)}p(t)\,dE(t)x,\ x\in D$

for all polynomials p, where $E(\cdot)$ is a generalized spectral measure on D supported by $\sigma(T)$ and commuting with any $V\in B(X)$ which commutes with T.

This decomposition is "maximal-unique," meaning that if W is an invariant linear manifold for T for which (a)–(d) are valid with W replacing D, then $W\subset D$ and the transformations S, N and $E(b)$ $(b\in B)$ corresponding to W are the restrictions to W of the respective transformations associated with D.

The proof uses a refinement of the method we applied in the proof of Theorem 3.13 in [3].

It turns out that $D=D_{k+2}$. For each $x\in D$, the map $f\to T(f)x$ of C^{k+2} into X has an extension as a continuous linear map of C^k into D given by

$$T(f)x = \sum_{j\leq k}(1/j!)\int_{\sigma(T)} f^{(j)}(t)\,dE(t)N^jx$$

(for all $f\in C^k$ and each $x\in D$). The extended map $f\to T(f)$ of C^k into $T(D)$ is multiplicative.

Keeping in mind the usual definition of a resolution of the identity, it is interesting to notice that if N (or S) is closable, then $E(b)$ commutes with S and N and $E(a\cap b)=E(a)E(b)$ for all a, $b\in B$. This is true in particular if $k=0$, since $N=0$ (cf. (c)) is trivially closable.

Theorem 2 may be given a version fitting into Dunford's theory of spectral operators [1]. Since $D=D_{k+2}$, D is a normed linear space under the norm $\|x\|=|x|_{k+2,T}$. Let us call its completion Y the Jordan space for T. T induces in a natural way an operator $T_Y\in B(Y)$.

THEOREM 2′. *Let T be as in Theorem 2 (with X not necessarily reflexive). Then $(T_Y)^*$ is spectral of class Y and type k.*

The case $k=0$ has a distinguished position if X is a Hilbert space. By Theorem 5 in [2], Condition (1) by itself is then sufficient for T to be spectral of scalar type. This is no longer true (in Hilbert space) for $k\geqq 1$, even when $\sigma(T)$ is a sequence with 0 as its only limit point. In Banach space (even reflexive) this breaks down even for $k=0$ (cf. [2, p. 176]). Let $P(R)$ denote the ring of polynomials over R. Condition (1) for $k=0$ is equivalent to the condition $|e^{ip(T)}|<M<\infty$ for all $p\in P(R)$ of degree $\leqq 1$. Dropping this limitation on the degree, we get a criterion for spectrality which is valid in any weakly complete Banach space.

THEOREM 3. *$T\in B(X)$ is of class C and has real spectrum if and only if*

$$\sup_{p\in P(R)} |e^{ip(T)}| < \infty. \tag{2}$$

If X is weakly complete, Condition (2) is necessary and sufficient for T to be spectral of scalar type with real spectrum.

The proof uses Theorem 2 in [4].

REFERENCES

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