ON INFINITE INSEPARABLE EXTENSIONS OF EXPONENT ONE

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Let K be a field of characteristic $p \neq 0$ and Der K denote the vector space over K of all derivations of K. A classical theorem of Jacobson [2], strengthened by the author [1], asserts that the subfields L of K with $L \supset K^p$ and $[L:K]$ finite are in natural one-one correspondence with the finite dimensional "restricted" subspaces of Der K, i.e., with those subspaces V such that $\dim_K V < \infty$ and such that $\phi \in V$ implies $\phi^p \in V$; the correspondence associates to L the space $\mathrm{Der}_L\, K$ of all derivations vanishing on L. (It follows that a finite dimensional restricted subspace is necessarily a Lie algebra.) The problem of extending this result after the fashion of Krull to fields $L \supset K^p$ with $[K:L]$ possibly infinite has been raised explicitly (cf. [3, p. 191]) but not answered. The purpose of this note is to show that the obvious conjecture in fact holds.

1. The Krull topology and statement of the main theorem. Let Der K be topologized by taking as a base for the neighborhoods of zero those subspaces V of the form $\mathrm{Der}_L\, K$ with L a finite extension $K^p(x_1, \cdots, x_n)$ of K^p; this will be called the *Krull topology*. The closure of an arbitrary subspace V in the Krull topology will be denoted by $\overline{V}$. Given an arbitrary element ϕ of Der K, the set of all $x \in K$ which are constants for ϕ, i.e., such that $\phi(x) = 0$, will be denoted K_ϕ. We shall further denote by D_ϕ the smallest restricted subspace of Der K containing ϕ, and by $\overline{D}_\phi$ its closure.

It is immediate that the closure of a restricted subspace is again restricted, and that a subspace of the form $\mathrm{Der}_L\, K$ is both closed and restricted.

THEOREM. *Let K be a field of characteristic $p \neq 0$. Then the subfields L containing K^p are in natural one-one correspondence with the closed restricted subspaces of* Der K, *the correspondence assigning to L the space* $\mathrm{Der}_L K$. *(It follows that a closed restricted subspace is in particular a Lie algebra.) Further, every closed restricted subspace is of the form $\overline{D}_\phi$ for some ϕ in* Der K.

2. Proof of the theorem. Before the proof we give several lemmas.

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Throughout, K will denote the fixed field of characteristic p.

LEMMA 1. *Let* $\{K_\alpha\}_{\alpha\epsilon I}$ *be a collection of subfields of* K *such that* (i)$K_\alpha \supset K^p$ *for all* α, *and* (ii) *for all* α *and* β *in* I *there is a* γ *such that* $K_\gamma \subset (K_\alpha \cap K_B)$. *Set* $k = \bigcap K_\alpha$, *and let* $x_1, \cdots, x_n$ *be a finite set of elements of* K *which are* p-*independent over* k (*i.e., such that the monomials* $x_1^{i_1} \cdots x_n^{i_n}$, $0 \leqq i_q < p$ *are linearly independent*; *cf.* [5, *p.* 129]). *Then* $x_1, \cdots, x_n$ *are already* p-*independent over* K_{α_0} *for some* α_0.

PROOF. The proof for $n=1$ is trivial, since condition (i) on the K_α implies that given α, either x_1 is p-independent over K_α or x_1 is in K_α; since $x_1 \notin \bigcap K_\alpha$, it follows that for some α_0, $x_1 \notin K_{\alpha_0}$. The proof now proceeds by induction. Suppose for some α_1 that $x_1, \cdots, x_{n-1}$ are p-independent over k_{α_1}. replacing every K_α by $K_\alpha \cap K_{\alpha_1}$, we may assume, without loss of generality, that $x_1, \cdots, x_{n-1}$ are p-independent over K_α for all α. We must show that for some α_0 we have $x_n \notin K_{\alpha_0}(x_1, \cdots, x_{n-1})$. But $K_\alpha(x_1, \cdots, x_{n-1})$ is naturally isomorphic to $K_\alpha \otimes_k k(x_1, \cdots, x_{n-1})$ for all α, whence $\bigcap K_\alpha(x_1, \cdots, x_{n-1}) = (\bigcap K_\alpha) \otimes_k k(x_1, \cdots, x_{n-1}) = k \otimes_k (x_1, \cdots, x_{n-1}) = k(x_1, \cdots, x_{n-1})$. Since the latter does not contain x_n, it follows that $x_n \notin K_{\alpha_0}(x_1, \cdots, x_{n-1})$ for some α_0, as required. This completes the induction and the proof.

The following is essentially contained in [1, bottom of p. 563].

LEMMA 2. *Let* $K^p \subset k$, $\{x_\alpha\}$ *be a* p-*basis for* K *over* k, *and* ϕ *be an element of* $\text{Der}_k K$ *such that* $\phi(x_\alpha) = x_\alpha^{p+1}$, *all* α. *Then* $K_\phi = k$.

PROOF. It is sufficient to show that if M is any monomial of the form $M = x_{\alpha_1}^{j_1} \cdots x_{\alpha_n}^{j_n}$, $0 \leqq j_q < p$, then $\phi(M) = 0$ implies $M = 1$. Set $x_\alpha^p = \lambda_\alpha$. Then $\phi(M) = \lambda_{\alpha_1} j_1 + \cdots + \lambda_{\alpha_n} j_n$. Since the x_α are p-independent over k, the λ_α are surely linearly independent over the prime field. Therefore $\phi(M) = 0$ if and only if $j_1 = \cdots = 0$, i.e., if and only if $M = 1$. This ends the proof.

LEMMA 3. *Let the elements of* Der K *be partially ordered by setting* $\phi \succ \phi'$ *if* $K_\phi \subset K_{\phi'}$. *Suppose* V *a closed and restricted subspace of* Der K, *and let* $\{\phi_\alpha\}_{\alpha \in I}$ *be a linearly ordered subset of* V. *Set* $k = \bigcap K_{\phi_\alpha}$. *Then* (1) $V \supset \text{Der}_k K$ *and* (2) *there exists a* $\phi \in V$ *such that* $K_\phi = k$. *This* ϕ *is then an upper bound in the partial order for the* ϕ_α, *whence by Zorn's lemma,* V *contains a maximal element.*

PROOF. Since V is closed, to show that $V \supset \text{Der}_k K$, it is sufficient to show that if ϕ is in $\text{Der}_k K$ and if $x_1, \cdots, x_n$ are arbitrary elements of K in finite number, then there exists a $\theta \in V$ with $\theta(x_i) = \phi(x_i)$ $i = 1, \cdots, n$. Without loss of generality, we may assume that for some m, $x_1, \cdots, x_m$ are p-independent over k and that $x_{m+1}, \cdots, x_n \in k(x_1, \cdots, x_m)$. By Lemma 1, there exists an α such that $x_1, \cdots, x_m$ are p-independent over K_{ϕ_α} and by Lemma 2 of [1], there exist in D_{ϕ_α}—and hence in V—derivations $\phi_1, \cdots, \phi_m$ such that $\phi_i(x_j) = \delta_{ij}$, $i, j = 1, \cdots, m$. It follows that D_{ϕ_α} contains a θ taking on arbitrary values on $x_1, \cdots, x_m$, whence in particular, such that $\theta(x_i) = \phi(x_i)$, $i = 1, \cdots, m$. Since $K_{\phi_\alpha} \supset k$, it follows that $\theta(x_i) = \phi(x_i)$ for $i = m+1, \cdots, n$ as well, proving (1). It follows from Lemma 2 that $\mathrm{Der}_k K$ contains a ϕ such that $K_\phi = k$. This ends the proof.

LEMMA 4. *Let V be a closed restricted subspace of* $\mathrm{Der}_k K$, *and suppose given $\phi \in V$, $y \in K$ such that $\phi(y) = 1$. Let $\{y, x_\alpha\}_{\alpha \in I}$ be a p-basis of K over K^p. Then the derivation θ defined by $\theta(y) = 1$, $\theta(x_\alpha) = 0$, all $\alpha \in I$, is in V.*

PROOF. By Lemma 2 of [1], for any finite subset $x_{\alpha_1}, \cdots, x_{\alpha_n}$ of $\{x_\alpha\}$, if we write $y = x_{\alpha_1}$, there exist $\phi_1, \cdots, \phi_n$ such that $\phi_i(x_{\alpha_j}) = \delta_{ij}$; in particular, $\phi_1(y) = 1$, $\phi_1(x_{\alpha_i}) = 0$, $i = 2, \cdots, n$. It follows that for any finite extension L of K^p contained in K there exists a ϕ_1 in V coinciding with θ on L. Since V is closed, it follows that θ is in V.

We come now to the

PROOF OF THE THEOREM. Let V be a closed restricted subspace of Der K, ϕ be a maximal element of V, and set $K_\phi = L$. Then $V \supset \mathrm{Der}_L K$. Suppose, if possible, that $V \neq \mathrm{Der}_L K$. Then there exists a $y \in L$ and $\psi \in V$ such that $\psi(y) \neq 0$; we may suppose that $\psi(y) = 1$. Let $\{x_\alpha\}$ be a p-basis of K over L and $\{y, z_\beta\}$ be a p-basis of L over K^p. Then V contains a θ, by Lemma 4, such that $\theta(y) = 1$, $\theta(x_\alpha) = \theta(z_\beta) = 0$, all α, β. Set $L' = K^p(\{z_\beta\})$, so that $L'(y) = L$. Then $\theta, \phi \in \mathrm{Der}_{L'} K$. Let ω be the element of $\mathrm{Der}_L K$ defined by $\omega(x_\alpha) = x_\alpha^{p+1}$, all α. Then $\omega + y^{p+1}\theta$ has L' as its field of constants by Lemma 3, contradicting the maximality of ϕ. It follows that $V = \mathrm{Der}_{K_\phi} K$ for any maximal ϕ in V. Since it is trivial that any subspace of Der K of the form $\mathrm{Der}_L K$ is closed and restricted, it follows that $L \to \mathrm{Der}_L K$ is a one-one correspondence between those subfields L of K with $K^p \subset L$ and the closed restricted subspaces of Der K. Finally, observe that if ϕ is maximal in V, then $\overline{D}_\phi$ is also closed, restricted, and therefore coincides with $\mathrm{Der}_{K_\phi} K = V$. This ends the proof.

3. **p-convexity (Shimura-Ponomarenko).** If x is an arbitrary element of K, then we shall denote by H_x the set of all ϕ in Der K such that $\phi(x) = 0$; H_x is the "hyperplane" in Der K determined by x and is an open set in the Krull topology. Following a suggestion of Shimura, a subspace V of Der K has been called *p-convex* by Pono-

marenko if $\bigcap(V+H_x)=V$, the intersection being taken over all $x\in K$. For any subspace V of Der K, we may defined the *p-hull* of V, denoted hull V, to be $\bigcap(V+H_x)$. An element ϕ of Der K is then in hull V if and only if for every x its H_x-neighborhood, $\phi+H_x$, meets V. It follows that hull $V\supset\overline{V}$. Since, as may be readily seen, hull(hull V) $=$hull V, it follows that hull V is closed in the Krull topology.

Ponomarenko [4] has proved that a necessary and sufficient condition that a subspace V of Der K be of the form $\text{Der}_L K$ for some subfield L of K containing K^p is that V be p-convex, i.e., that V be its own hull. While Ponomarenko's result is, as he shows, a simple and direct consequence of the work of Jacobson, it may also be of interest to observe that the result follows immediately from our main theorem. Indeed, all that need be shown is that if V is p-convex then V is restricted. To this end observe that if V is p-convex then $\phi\in V$ and $\phi\succ\phi'$ imply $\phi'\in V$. Now for any x in K, define an element ϕ_x of V by setting $\phi_x=\phi$ if $\phi(x)=0$ and $\phi_x=\phi'(x)\phi(x)^{-1}\phi$ otherwise. Since by definition $\phi\succ\phi'$ if and only if $\phi(x)=0$ implies $\phi'(x)=0$ for all x, it follows that ϕ_x is in the H_x-neighborhood of ϕ' for all x, showing that $\phi'\in$hull $V=V$, as required. Finally, for any ϕ we have $\phi\succ\phi^p$, showing that V p-convex implies V restricted, as asserted.

If we define a subspace V of Der K to be a *lattice* if $\phi\in V$ and $\phi\succ\phi'$ imply $\phi'\in V$, then we have in fact observed the following trivial implications: V p-convex$\Rightarrow V$ closed, lattice$\Rightarrow V$ closed, restricted. Since our main theorem implies that a closed restricted V is of the form $\text{Der}_L K$, and since any subspace of the latter form is trivially p-convex, it follows that the implications are all equivalences.

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