THE COHOMOLOGY OF CLASSIFYING SPACES OF H-SPACES

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Let G denote an associative H-space with unit (e.g. a topological group). We will show that the relations between G and a classifying space B_G are more readily displayed using a geometric analog of the resolutions of homological algebra. The analogy is quite sharp, the stages of the resolution, whose base is B_G, determine a filtration of B_G. The resulting spectral sequence for cohomology is independent of the choice of the resolution, it converges to $H^*(B_G)$, and its E_2-term is $\mathrm{Ext}_{H(G)}(R, R)$ (R=ground ring). We thus obtain spectral sequences of the Eilenberg-Moore type [5] in a simpler and more geometric manner.

1. **Geometric resolutions.** We shall restrict ourselves to the category of compactly generated spaces. Such a space is Hausdorff and each subset which meets every compact set in a closed set is itself closed (a k-space in the terminology of Kelley [3, p. 230]). Subspaces are usually required to be closed, and to be deformation retracts of neighborhoods.

Let G be an associative H-space with unit e. A right G-action on a space X will be a continuous map $X\times G\rightarrow X$ with $xe=x$, $x(g_1g_2)=(xg_1)g_2$ for all $x\in X$, $g_1, g_2\in G$. A space X with a right G-action will be called a G-space. A G-space X and a sequence of G-invariant closed subspaces $X_0\subset X_1\subset \cdots \subset X_n\subset \cdots$ such that $X_0\neq \emptyset$, $X=\bigcup_{i=0}^{\infty} X_i$, and X has the weak topology induced by $\{X_i\}$ will be called a *filtered* G-space.

1.1. DEFINITION. (a) A filtered G-space X is called *acyclic* if for some point $x_0\in X_0$, X_n is contractible to x_0 in X_{n+1} for every n.

(b) A filtered G-space X is called *free* if, for each n, there exists a closed subspace D_n ($X_{n-1}\subset D_n\subset X_n$) such that the action mapping $(D_n, X_{n-1})\times G\rightarrow(X_n, X_{n-1})$ is a relative homeomorphism.

(c) A filtered G-space X is called a *G-resolution* if X is both free and acyclic.

Under the restrictions we have imposed on subspaces, the acyclicity condition implies that X is contractible.

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1.2. THEOREM. *If G is a topological group, any G-resolution X is a principal G-bundle over $B_G=X/G$ with action $X\times G\to X$ as principal map.*

When G is a topological group, Milnor's construction [4], where X_n is the join of $n+1$ copies of G, is a G-resolution. In the general case, the existence of a G-resolution is given by the Dold-Lashof construction [2].

There is also a comparison theorem. Let G, G' be H-spaces, $\Phi: G\to G'$ a morphism, X, X' filtered G, G'-spaces. An *extension* Φ' of Φ is a map $\Phi': X\to X'$ with $\Phi'(X_n)\subset X'_n$ and $\Phi'(xg)=\Phi'(x)\Phi(g)$. If Φ', Φ'' are two extensions of Φ, a *homotopy* h will be a map $h: X\times I\to X'$ with $h_0=\Phi'$, $h_1=\Phi''$, $h(X_n\times I)\subset X'_{n+1}$, and $h(xg, t)=h(x, t)\Phi(g)$.

1.3. MAPPING THEOREM. *If $\Phi: G\to G'$ is a morphism, X a free filtered G-space, X' an acyclic filtered G'-space, then Φ has an extension Φ': $X\to X'$. Furthermore, any two such extensions are homotopic.*

Thus in particular, for any two resolutions X, X' of G there exists an equivariant $\mu: X\to X'$, unique up to equivariant homotopy.

We define the product of two filtered spaces X, X' to be the product space $X\times X'$ filtered by $(X\times X')_n=\bigcup_{i=0}^{n} X_i\times X_{n-i}$.

1.4. THEOREM. *If X is a G-resolution and X' a G'-resolution, then $X\times X'$ is a $G\times G'$-resolution.*

2. **The spectral sequence.** When X is a G-resolution, let $B=X/G$ denote the decomposition space by maximal orbits, let $p: X\to B$ be the projection and $B_n=p(X_n)$. If R is a coefficient ring, the filtration $\{B_n\}$ of B determines two spectral sequences, the homology spectral sequence $E_*(B, R)=\{E^r, d_r\}$ and the cohomology spectral sequence $E^*(B, R)=\{E_r, d^r\}$.

2.1. THEOREM. (a) *The spectral sequences E_*, E^* are functors from the category of H-spaces and continuous morphisms to the category of bigraded spectral sequences. (We regard all spectral sequences as beginning with E^2, E_2.)*

(b) *If the homology algebra $H(G)=H(G;\ R)$ is R-free, then as a bigraded R-module*

$$E^2\cong \operatorname{Tor}^{H(G)}(R, R),\qquad E_2\cong \operatorname{Ext}_{H(G)}(R, R).$$

(c) $E_*\Rightarrow H(B; R)$. *If R is compact or $H(G)$ is free then $E^*\Rightarrow H^*(B; R)$.*

Proposition (a) follows from 1.3, (c) is true in any filtered space, and (b) is proved using the Milnor-Dold-Lashof construction, in fact the E^1-term in this case is precisely the bar resolution of R over the algebra $H(G)$.

In order to deepen these results to include products, we develop the theory of $\times$-products for the spectral sequences of filtered spaces X, Y. These are natural transformations μ: $E^r(X) \otimes E^r(Y) \to E^r(X \times Y)$, ν: $E_r(X) \otimes E_r(Y) \to E_r(X \times Y)$ which behave nicely with respect to differentials. They are isomorphisms when R is a field and $E_1(X)$ is of finite type.

The diagonal morphism Δ: $G \to G \times G$ induces, by 2.1(a), a mapping of the cohomology spectral sequences Δ^*: $E_r(B_G \times B_G) \to E_r(B_G)$. Composing Δ^* with ν (where $X = Y = B_G$) gives the multiplication in E_r.

2.2. THEOREM. *With respect to this multiplication, $E_r(B_G)$ is a commutative, associative, bigraded, differential algebra with unit. The multiplication on E_{r+1} is induced by that on E_r. The multiplications commute with the convergence* 2.1(c). *When $H(G)$ is R-free, the second isomorphism of* 2.1(b) *preserves products.*

When R is a field, the composition $\mu^{-1}\Delta_*$ defines a co-algebra structure in the homology spectral sequence having dual properties.

3. **Co-algebra structure.** We assume in this section that R is a field and $H(G)$ is of finite type. When G is commutative the multiplication m: $G \times G \to G$ is also a morphism. Then the composition $m_*\mu$ gives an algebra structure on E_*, and $\nu^{-1}m^*$ a co-algebra structure in E^*. Actually the same is true if G is the loop space of an H-space. This yields

3.1. THEOREM. *If G is commutative or the loop space of an H-space, then E_r, E^r are bicommutative, biassociative, differential, bigraded Hopf algebras with (E^r, d_r) the dual algebra to (E_r, d^r). The Hopf algebra structure on $E_2 = \mathrm{Ext}_{H(G)}(R, R)$ is the natural one arising from the Hopf algebra structure on $H(G)$. Moreover if G is connected and R is perfect, then E_r is primitively generated on elements of bi-degree $(1, q)$, $(2, q')$, and $d^r = 0$ except for $r = p^k - 1$ or $2p^k - 1$ where $p = \mathrm{Char}\ R$. If $G = \Omega(H)$, H homotopy associative, then $E_\infty \approx H^*(B; R)$ as an algebra.*

Actually one can give an explicit description of E_{r+1} in terms of E_r and $d^r(x^{1,q})$, $d^r(x^{2,q'})$, where $x^{1,q}$, $x^{2,q'}$ are primitive generators.

4. **Applications.** Moore pointed out [5] that his spectral sequence gives an easy proof of the theorem of Borel which states: *If $H(G)$ is an exterior algebra with generators of odd dimensions and is R-free, then $H^*(B_G)$ is a polynomial algebra on corresponding generators of one higher dimension.* Moore argues that a brief computation shows that

the E_2-term, $\mathrm{Ext}_{H(G)}(R, R)$, is just such a polynomial algebra. Then all terms of E_2 of odd total degree are zero. Hence every $d^r=0$, so $E_2=E_\infty$. Since E_∞ is a polynomial algebra, it is algebraically free; and therefore $H^*(B_G) \approx E_\infty$ as an algebra.

An Eilenberg-MacLane space of type (π, n) can be realized by a commutative topological group G, and its B_G is of type $(\pi, n+1)$. Consequently $H(\pi, n)$ and $H^*(\pi, n+1)$ are connected by a spectral sequence of Hopf algebras $E_r(B_G)$.

4.1. THEOREM. *If G is of type (π, n), π is a finitely generated abelian group, and $R=Z_p$ where p is a prime, then the spectral sequence collapses*

$$\mathrm{Ext}_{H(G)}(Z_p, Z_p) \approx E_2 = E_\infty \approx H^*(B_G).$$

This implies that $H^*(\pi, n; Z_p)$ is a free commutative algebra for every n. In fact an algorithm is obtained for computing $H^*(\pi, n; Z_p)$ as a primitively generated Hopf algebra over the algebra of reduced pth powers. These results confirm and amplify results of H. Cartan.

For another application, let K be a compact, simply-connected Lie group, and let G be the loop space of K. Using Bott's result [1] that $H(G; Z)$ is torsion free, we obtain

4.2. THEOREM. (a) *If $p>5$, the spectral sequence collapses*

$$\mathrm{Ext}_{H(G)}(Z_p, Z_p) \approx E_2 = E_\infty \approx H^*(K; Z_p) \approx \Lambda(x_1, \cdots, x_r)$$

where $x_1, \cdots, x_r$ are generators of the dimensions of the primitive invariants of K. In particular K has no p-torsion, and $H^*(K; Z_p) \approx H^*(K; Z) \otimes Z_p$.

(b) *If $p=3$ or 5, there is at most one nonzero differential, namely, d^{2p-1}. Moreover $H^*(K; Z_p)$ and $H_*(G; Z_p)$ can be constructed explicitly from the Betti numbers of K and the dimensions of the kernels of the maps $x \to x^p$ and $x \to x^{p^2}$ where $x \in H^2(G; Z_p)$.*

(c) *For any $p>2$, we have $u^p=0$ for all $u \in \tilde{H}^*(K; Z_p)$.*

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