RESEARCH PROBLEMS

18. Fred Gross: *Function theory*

Let S be an arbitrary region. Does there exist a transcendental meromorphic function with the property that the pre-image $f^{-1}(S)$ is of finite measure? (Received July 16, 1965.)

19. Fred Gross: *Function theory*

In Volume 4 of the Michigan Mathematical Journal Paul Erdös asked the following question:

If A and B are two denumerable dense sets, does there exist an entire function which maps A onto B?

This problem is quite difficult. One can ask, however, a simpler question.

Do there exist two dense denumerable sets A and B, such that, $f(z)\in A$ if and only if $z\in B$ implies that $f(z)$ is linear? More generally, do there exist two linearly independent functions $f(z)$ and $g(z)$ such that $f(z)\in A$ if and only if $g(z)\in B$. (Received July 16, 1965.)

20. Albert A. Mullin: *Stochastic number-theory*

The probability that a random natural number has the prime factor p is $1/p$. Hence the probability that two random natural numbers have the common prime factor p is $1/p^2$. Thus, the probability that two random natural numbers have no common prime factor (i.e., the probability that they are relatively prime) is $\prod_p (1-1/p^2) =1/\zeta(2)=6/\pi^2$, by Euler's Identity (an *analytic* version of unique factorization. What is the probability that two random natural numbers satisfy the condition that their *mosaics* have no prime in common? Clearly this measure is strictly positive but it is bounded above by $6/\pi^2$. To what extent is statistical *dependence* crucial to the argument? What is the probability that a random natural number has only odd primes in its mosaic? (Received July 16, 1965.)

21. Richard Bellman: *Differential equations*

The equation $u'(t)=(a+bu(t_1))u(t)$, $u(0)=c$, valid for $0\leqq t\leqq t_2$, with $t_2>t_1$, shows that an equation of the form $u'(t)=g(u(t), u(t_1), u(t_2), \cdots, u(t_N))$, $u(0)=c$, $0<t_1<t_2<\cdots<t_N$, can have an infinite number of complex solutions and more than one real solution. What additional conditions on the solution ensure uniqueness of a real solution? (Received July 16, 1965.)

22. Richard Bellman: *Partial differential equations*

Mathematical models of respiratory control processes lead to partial differential equations of the form

$$u_t + g(u(x, t), u(x_1, t), \cdots, u(x_N, t))u_x = h(x, t),$$

where the equation holds for $t \geqq 0$, $-\infty < x < \infty$, and $u(x, 0)$ is prescribed. Under what conditions can one assert that a unique solution exists?

Consider similar problems for the heat equation. (Received July 16, 1965.)

23. A. H. Stroud: *Factorization of Hankel matrices*

Find a Jacobi-like algorithm for factoring an arbitrary real Hankel matrix C

$$C = \begin{bmatrix} c_0 & c_1 \cdots c_{n-1} \\ c_1 & c_2 \cdots c_n \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ c_{n-1} & c_n \cdots c_{2n-2} \end{bmatrix}$$

in the form

$$X^T A X = C \tag{1}$$

where X is a Vandermonde matrix and A is diagonal

$$X = \begin{bmatrix} 1 & x_1 \cdots x_1^{n-1} \\ 1 & x_2 \cdots x_2^{n-1} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ 1 & x_n \cdots x_n^{n-1} \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 & 0 & \cdots 0 \\ 0 & A_2 & \cdots 0 \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ 0 & 0 & \cdots A_n \end{bmatrix}.$$

The solution of this problem would be a method for determining a quadrature formula

$$\int_a^b w(x)f(x)\,dx \cong \sum_{i=1}^n A_i f(x_i) \tag{2}$$

which is exact for all polynomials of degree $\leqq 2n-2$, provided we set

$$c_i = \int_a^b w(x)x^i\,dx, \qquad i = 0, 1, \cdots, 2n-2.$$

If det $C \neq 0$ then there is a one parameter family of quadrature formulas (2) and hence of factorizations (1). Quadrature formulas (2) can be constructed using orthogonal polynomials. (Received July 16, 1965.)

24. ANALYTIC FUNCTION THEORY

A Symposium on Analytic Function Theory was held at the University of Kentucky, May 28–June 1, 1965, with support from the University of Kentucky Research Fund.

The following is a list of problems drawn up by members of the Symposium.

A. Let $\mathcal{A}$, H^∞, and A denote respectively the space of all analytic functions defined on the unit disc D, the space of bounded analytic functions on D and the space of analytic functions on D that extend continuously to $\overline{D}$. Can any one of these be decomposed into the direct sum of two nontrivial closed subalgebras B_1 and B_2? For example, in the case of A, could $B_1 = \{f(z^2) \mid f \in A\}$?

B. Let $C-$ be the closure in L^∞ (of the circle with Lebesgue measure) of those functions whose negative Fourier coefficients are zero, except for possibly a finite number. For example, continuous functions, boundary values of functions in H^∞, and continuous functions times functions in H^∞ are in $C-$. How can the class $C-$ be characterized?

C. Let $\mathfrak{M}$ denote a closed subalgebra of H^∞. To say that $\mathfrak{M}$ is an *intrinsic* algebra means that $f \circ \phi$ is in $\mathfrak{M}$ whenever f is in $\mathfrak{M}$ and ϕ is a conformal one-to-one map of the unit disc onto itself (ϕ is a linear fractional transformation). Classify the intrinsic subalgebras of H^∞. Is the algebra A, (see problem A) a maximal intrinsic algebra? The algebra generated by the Blaschke products is an intrinsic algebra. It is H^∞?

D. The following theorem is due to D. J. Newman.

THEOREM. *If f_n is a sequence of functions in H^1 (of the circle) and f is in H^1 such that*
(a) $\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1$, *and*
(b) *f is the weak limit of f_n; then*

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0.$$

Is there an analogue of this result for $H^1(dm)$ of a Dirichlet algebra?

E. Let Ω be a simply connected region that contains the upper half plane and has 0 as a boundary point. Let f be the conformal map of the upper half plane onto Ω such that $f(i)=i$ and $f(0)=0$. What conditions are needed on the boundary of Ω in order that there be an inequality of the form $|f(z)|<c|z|^{1-\delta}$ with small positive δ, valid for $|z|<1$, $\operatorname{Im} z\geqq 0$?

F. A bounded holomorphic function on $|z|<1$ can be approximated by sums $R_n(z)=\sum_1^n (z_{n,k}-z)^{-1}$ with $|z_{n,k}|=1$ for $k=1,2,\cdots,n$. Can this be done in such a way that $(1-|z|)R_n(z)$ converges boundedly on the disc to $(1-|z|)f(z)$ as n approaches ∞?

G. Let Γ be a Jordan arc on the Riemann sphere S. Denote by A_Γ the class of functions continuous on S and analytic on $S-\Gamma$.

(a) Let f be a nonconstant member of A_Γ such that $f(\infty)=0$. Must f have infinitely many zeros on $S-\Gamma$? It is known that f vanishes somewhere on Γ.

(b) Does there exists a pair of functions f and g in A_Γ such that f and g together separate points on S? It is known that there exists a set of three such functions with this property.

H. An entire function $F(z)=\sum_0^\infty a_n z^{pn}=\phi(z^p)$, where p is an integer and the order of F is ρ, must have many of the properties of a function of order ρ/p. For example, if $p>2\rho$, F can have no asymptotic paths. If $p>\rho$, then the minimum modulus result for ϕ gives

$$\limsup\{\log m(r)/\log M(r)\} \geqq \cos(\pi\rho/p).$$

The growth indicator $h(\theta)$ must reach its maximum in any angle of opening π/p.

How far can these results be extended to entire functions $\sum_0^\infty a_n z^{\lambda_n}$ in which the sequence of integers λ_n has density less than $1/p$ in some sense?

I. If f is a continuous function from the unit circle C to the complex numbers, does there exist a homeomorphism ϕ from C to C such that $f\circ\phi$ has a uniformly convergent Fourier series? This is known to be true if f is real valued, or if $f(C)$ is a Jordan curve.

J. Is the space of all Keldyš-Lavrentieff maps connected, i.e. the set of all univalent functions f from the unit disc D such that $|f'|=1$ a.e. on the boundary of D?

K. Find conditions on subsets A and B of the unit disc D so that there will exist a holomorphic function ϕ from D to D so that $\phi(A)$ is near zero and $\phi(B)$ is near the boundary of D.

L. Suppose U is the unilateral shift operator on H^2 (of the unit circle). What are the cyclic vectors of U^*?

M. Let S denote the set of all schlicht functions on the unit disc D. If $f \in S$ and $g(z) = \int_0^z (f'(t))^\epsilon dt$ where ϵ is in $(0, 1)$, does this imply $g \in S$? The answer is known to be yes for $0 < \epsilon < (\sqrt{5} - 2)/3$.

N. If f is analytic in the unit disc D, and $|f(re^{i\theta})| < 1/(1-r^2)$, for what positive real numbers α is it true that $|f'(re^{i\theta})| < \alpha/(1-r^2)^2$? It is known that $\alpha = 4$ will suffice. What is the coefficient region for this class?

O. Let $C = \{z: |z| = 1\}$ and $L_2^{0+}(C)$ be the set of all $q \times q$ matrix valued functions Ψ on C such that $\Psi \in L_2(C)$ and for all $k < 0$, $\int_0^{2\pi} e^{-ki\theta} \Psi(e^{i\theta}) d\theta = 0$. Write $A > B$ to mean that $A - B$ is a $q \times q$ nonnegative hermitian matrix. It is known that, if

(1) $F > 0$ a.e. on C, $F \in L_1(C)$, and log det. $F \in L_1(C)$, then

(2) $F = \Phi\Phi^*$ a.e. on C, where $\Phi \in L_2^{0+}(C)$.

It is also known that in (2) we may take Φ to be optimal (or "outer"), i.e. satisfy the condition:

(3) if $\Psi_+ \in L_2^{0+}(C)$ and $\Psi_+\Psi_+^* = F$ a.e. then $\sqrt{\{\Psi_+(0)\Psi_+^*(0)\}} < \Phi_+(0)$, where Ψ_+ denotes the holomorphic extension of $\Psi \in L_2^{0+}(C)$ to the disc $[z: |z| < 1]$, and $\sqrt{}$ is the non-negative hermitian square root.

In case $q = 1$ we know

$$(4)\quad \Phi_+(z) = \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log F(e^{i\theta})\, d\theta, \quad |z| < 1.$$

The problem is to obtain a generalization of (4), i.e. a (closed form) expression for $\Phi_+(z)$ in terms of $F(e^{i\theta})$ valid for all $q \geqq 1$.

P. Let R be a bounded simply connected region in the complex plane. For each n, there exist "equilibrium points" $z_{n,1}, z_{n,2}, \cdots, z_{n,n}$ on the boundary of R with the property that $\prod_{j \neq k} |z_{n,j} - z_{n,k}|$ is an absolute maximum. Is it true that electrons at such points $z_{n,k}$ produce a small field in R, in the sense that, as $n \to \infty$

$$\sum_{k=1}^{n} \frac{1}{z_{n,k} - z} \to 0$$

uniformly on every compact subset of R? The answer is yes, if R is a Jordan region. (Received July 28, 1965.)