NONLINEAR MONOTONE OPERATORS AND CONVEX SETS IN BANACH SPACES

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Introduction. Let X be a real Banach space, X^* its conjugate space, (w, u) the pairing between w in X^* and u in X. If C is a closed convex subset of X, a mapping T of C into X^* is said to be monotone if

$$(Tu - Tv, u - v) \geqq 0 \tag{1}$$

for all u and v in C.

It is the object of the present note to prove the following theorem:

THEOREM 1. *Let C be a closed convex subset of the reflexive Banach space X with $0 \in C$, T a monotone mapping of C into X^*. Suppose that T is continuous from line segments in C to the weak topology of X^* while $(Tu, u)/\|u\| \to +\infty$ as $\|u\| \to +\infty$.*

Then for each given element w_0 of X^*, there exists u_0 in C such that

$$(Tu_0 - w_0, u_0 - v) \leqq 0 \tag{2}$$

for all v in C.

If $C = X$, Theorem 1 asserts that $Tu_0 = w_0$ and reduces to a theorem on monotone operators proved independently by the writer [1] and G. J. Minty [9] and applied to nonlinear elliptic boundary value problems by the writer in [2], [3], and [6]. (See also Leray and Lions [7].) If $C = V$, a closed subspace of X, the conclusion of Theorem 1 is that $Tu_0 - w_0 \in V^\perp$, which yields a variant of the generalized form of the Beurling-Livingston theorem proved by the writer in [4] and [5]. The conclusion of Theorem 1 for $C = X$ was extended by the writer to classes of densely defined operators (see [6] for references) and in [5] to multivalued mappings.

It is easily shown that Theorem 1 generalizes and includes as a special case the following linear theorem of Stampacchia, which has been applied by the latter to the proof of the existence of capacitary potentials with respect to second-order linear elliptic equations with discontinuous coefficients:[2]

THEOREM 2. *Let H be a real Hilbert space, C a closed convex subset of H, $a(u, v)$ a bilinear form on H which is separately continuous in u*

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Added in proof. A result similar to Theorem 1 has recently been obtained jointly by Hartman and Stampacchia (in an as yet unpublished paper) who also give a very interesting application to existence theorems for second order nonlinear elliptic equations.

and v. Suppose that there exists a constant $c>0$ such that $a(u, u) \geqq c\|u\|^2$ for all u in H.

Then for each w_0 in H, there exists u_0 in C such that

$$a(u_0, u_0 - v) \leqq (w_0, u - v) \tag{3}$$

for all v in C.

1. We denote weak convergence by $\rightharpoonup$, strong convergence by $\rightarrow$.

LEMMA 1. *If $u_0 \in C$, u_0 is a solution of the inequality (2) if and only if*

$$(Tv - w_0, v - u_0) \geqq 0 \tag{4}$$

for all v in C.

PROOF OF LEMMA 1. If for a given u_0 in C and all v in C, we have $(Tu_0 - w_0, u_0 - v) \leqq 0$, then since

$$(Tu_0 - Tv, u_0 - v) \geqq 0$$

by monotonicity, it follows that

$$(Tv, u_0 - v) \leqq (Tu_0, u_0 - v) \leqq (w_0, u_0 - v),$$

i.e.,

$$(Tv - w_0, v - u_0) \leqq 0.$$

Conversely, suppose the inequality (4) holds for all v in C. Suppose $v_0 \in C$, and for $0 < t \leqq 1$, set

$$v_t = (1 - t)u_0 + tv_0.$$

Then $v_t \in C$, $v_t - u_0 = t(v_0 - u_0)$, and we have

$$0 \leqq (Tv_t - w_0, t(v_0 - u_0)) = t(Tv_t - w_0, v_0 - u_0).$$

Since $t > 0$ may be canceled, we have

$$(Tv_t - w_0, v_0 - u_0) \geqq 0.$$

If we let $t \rightarrow 0$ and use the weak continuity of T on segments in C, we have $Tv_t \rightharpoonup Tu_0$, and hence

$$(Tu_0 - w_0, u_0 - v_0) \leqq 0. \text{ q.e.d.}$$

DEFINITION. *Let $c(r) = \inf_{\|u\| = r} \{(Tu, u)/\|u\|\}$.* By the hypothesis of Theorem 1, $c(r) \rightarrow +\infty$ as $r \rightarrow +\infty$. We have

$$(Tu, u) \geqq c(\|u\|)\|u\|, \qquad u \in C.$$

LEMMA 2. *There exists a constant M which depends only upon the* *function* $c(r)$ *and on* $\|w_0\|$ *such that if* u_0 *is a solution of the inequality* (2), *then* $\|u_0\| \leqq M$.

PROOF OF LEMMA 2. If

$$(Tu_0 - w_0, u_0 - v) \leqq 0, \qquad v \in C,$$

we have since $0 \in C$,

$$c(\|u_0\|)\|u_0\| \leqq (Tu_0, u_0) \leqq (Tu_0 - w_0, u_0) + (w_0, u_0) \leqq \|w_0\| \cdot \|u_0\|.$$

Hence

$$c(\|u_0\|) \leqq \|w_0\|$$

and

$$\|u_0\| \leqq M(\|w_0\|, c(r)). \text{ q.e.d.}$$

DEFINITION. *If* $G \subset X \times X^*$, G *is said to be a monotone set if* $[u, w]$, $[u_1, w_1] \in G$ *implies that* $(w - w_1, u - u_1) \geqq 0$.

G is said to be maximal monotone if it is monotone and maximal in the monotone sets ordered by inclusion.

LEMMA 3. *Under the hypotheses of Theorem* 1, *suppose that* C *has* 0 *as an interior point and let* $G \subset X \times X^*$ *be given by*

$$G = \{[u, w] \mid u \in C, w = Tu + z, \text{ where } (z, u - v) \geqq 0 \text{ for all } v \text{ in } C\}.$$

Then G *is a maximal monotone set in* $X \times X^*$.

PROOF OF LEMMA 3. G is a monotone set since if $[u, w]$ and $[u_1, w_1] \in G$, with $w = Tu + z$, $w_1 = Tu_1 + z_1$, then

$$(w - w_1, u - u_1) = (Tu - Tu_1, u - u_1) + (z, u - u_1) + (z_1, u_1 - u) \geqq 0.$$

Suppose on the other hand that $[u_0, w_0] \in X \times X^*$ with

$$(w_0 - w, u_0 - u) \geqq 0$$

for all $[u, w]$ in G. We assert first that $u_0 \in C$. Otherwise, $u_0 = sv_0$ for some v_0 on the boundary of C with $s > 1$. Let $z_0 = 0$ be an element of X^* such that $(z_0, v_0 - v) \geqq 0$ for all v in C. Since 0 is an interior point of C, $(z_0, v_0) > 0$. For each $\lambda > 0$, $[v_0, Tv_0 + \lambda z_0]$ lies in G. Hence

$$0 \leqq (w_0 - Tv_0 - \lambda z_0, u_0 - v_0) = (s - 1)(w_0 - Tv_0 - \lambda z_0, v_0).$$

Cancelling $(s - 1) > 0$, we have

$$\lambda(z_0, v_0) \leqq (w_0, v_0) - (Tv_0, v_0),$$

which is a contradiction since $(z_0, v_0) > 0$ and λ is arbitrary. Hence $u_0 \in C$.

In addition, for each u in C, $[u, Tu]$ lies in G. Hence

$$(Tu - w_0, u - u_0) \geqq 0.$$

Applying Lemma 1, we have

$$(Tu_0 - w_0, u_0 - v) \leqq 0, \qquad v \in C.$$

Hence $Tu_0 - w_0 = -z$, where $(z, u_0 - v) \geqq 0$ for all v in C. Hence $w_0 = Tu_0 + z$, and $[u_0, w_0] \in G$. q.e.d.

LEMMA 4. *Theorem 1 holds if X is a finite dimensional Banach space F.*

PROOF OF LEMMA 4. We may suppose without loss of generality that $w_0 = 0$, that F is a finite dimensional Hilbert space with $F^* = F$, and that C spans F and hence has an interior point v_0 in F. Replacing C by $C_0 = v_0 - C$ and defining a new mapping T' on C_0 by $T'u = -T(v_0 - u)$, it is easy to verify that we may assume that 0 is an interior point of C and the condition on (Tu, u) is replaced by

$$(Tu, u - v_0) \geqq c(\|u\|)\|u\|$$

for a given v_0 in C, with $c(r) \to +\infty$ as $r \to +\infty$.

Let G be the maximal monotone set in $F \times F^*$ constructed in Lemma 3. Then nG is maximal monotone for each positive integer n. By a theorem of Minty [8], for each $n > 0$, there exists $[u_n, w_n] \in G$ such that

$$u_n + nw_n = 0.$$

Since $w_n = Tu_n + z_n$, where $(z_n, u_n - v) \geqq 0$ for all v in C, we have

$$\begin{aligned} -\left(\frac{1}{n} u_n, u_n - v_0\right) &= (w_n, u_n - v_0) \\ &= (Tu_n, u_n - v_0) + (z_n, u_n - v_0) \geqq c(\|u_n\|)\|u_n\|, \end{aligned}$$

while

$$-\left(\frac{1}{n} u_n, u_n - v_0\right) \leqq \frac{1}{n} \|u_n\| \cdot \|v_0\|.$$

Thus $c(\|u_n\|) \leqq n^{-1}\|v_0\|$, and $\|u_n\| \leqq M$, independent of n.

We may extract a subsequence which we again denote by u_n such that $u_n \to u_0$ in F. Then $w_n \to 0$. For each u in C

$$(Tu - w_n, u = u_n) \geqq 0.$$

Taking the limit as $n \to \infty$, we have

$$(Tu, u - u_0) \geqq 0, \qquad u \in C.$$

By Lemma 1,

$$(Tu_0, u_0 - v) \leqq 0$$

for all v in C. q.e.d.

PROOF OF THEOREM 1. It suffices to take $w_0 = 0$. For each finite dimensional subspace F of X, let $C_F = C \cap F$, j_F be the injection map of F into X, j_F^* the dual projection map of X^* onto F^*. We set

$$T_F = j_F^*(T \mid C_F) : C_F \to F^*.$$

Then T_F satisfies the hypotheses of Lemma 4, and there exists u_F in C_F such that

$$(T_F u_F, u_F - v) = (Tu_F, u_F - v) \leqq 0, \qquad v \in C_F.$$

By Lemma 2, since for u in C_F,

$$(T_F u, u) = (Tu, u) \geqq c(\|u\|)\|u\|,$$

there exists a constant M independent of F such that $\|u_F\| \leqq M$. Since X is reflexive and C is weakly closed, there exists u_0 in C such that for every finite dimensional F, u_0 lies in the weak closure of the set $V_F = \bigcup_{F \subset F_1} \{u_{F_1}\}$.

Let v be an arbitrary element of C, F a finite dimensional subspace of X which contains v. For u_{F_1} in V_F, by Lemma 1,

$$(Tv, v - u_{F_1}) \geqq 0.$$

Since $(Tv, v - v_1)$ is weakly continuous in v_1, we have

$$(Tv, v - u_0) \leqq 0, \qquad v \in C.$$

By Lemma 1, $(Tu_0, u_0 - v) \geqq 0$ for v in C. q.e.d.

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