FUNCTION SPACES[1]

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1. **Introduction.** What I shall say is directed towards the explicit description and study of individual functionals and operators. I first consider the function spaces $C_n^m(D)$, B, K, Z (defined below) and their adjoints. Then I consider the factorization of operators.

If X is a normed linear space, its *adjoint*, or conjugate, or dual, X^*, is defined as the space of linear continuous functionals on X, with norm

$$\|F\| = \sup_{x\in X;\|x\|=1} |Fx|, \qquad F \in X^*.$$

The space X^* is determined by X. For some X, our knowledge of X^* is complete and useful. This is the case if X is a Hilbert space, or an L^p-space, $p \geqq 1$, or the space $C_0(D)$ of continuous functions on a compact domain D [2, Chapter 4]. For some X, as we shall see, our knowledge of X^* is incomplete.

Definitive theorems about the spaces $C_n(I)^*$, B^*, K^*, and Z^* are given in §2 and §4. These theorems provide accessible standard forms for Fx, $x\in X$, and explicit procedures for calculating $\|F\|$, where $F\in X^*$ and X is $C_n(I)$, B, or Z. Theorem 6 provides an accessible form, free of Stieltjes integrals, for Fx, $x\in B$, where $F\in K^*$.

The theorems of §3 about $C_n^m(D)^*$ appear to be new. Theorem 2 asserts the existence of a standard form for Fx, $x\in C_n^m(D)$, where $F\in C_n^m(D)^*$. Theorems 3 and 4 describe the functional 0 as an element of $C_1^m(I)^*$ and $C_2^2(I)^*$.

Just as X determines X^*, so a pair X, Y of normed linear spaces determines the space $\mathfrak{J}(X, Y)$ of linear continuous operators on X to Y. If we wish to study an operator $T_0\in\mathfrak{J}(X, Y)$, the properties of T_0 common to all elements of $\mathfrak{J}(X, Y)$ may be insufficient to provide an accessible form for T_0x, $x\in X$. It is often useful to study T_0 as an individual and, if possible, to write T_0 as a product of linear continuous operators. Such factorizations and their use in the theory of approximation are considered in §5.

Theorem 10 is a dual of Fubini's theorem.

2. **The space** $C_n(I)$. Let I be a compact linear interval and n a nonnegative integer. The space $C_n(I)$ consists of functions on I which

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are continuous together with their derivatives of order $\leqq n$, with norm either

$$|||x||| = \max_{i=0,\cdots,n} \sup_{s\in I} |x_i(s)|, \qquad x \in C_n(I),$$

or

$$||x|| = \max[|x(a)|, |x_1(a)|, \cdots, |x_{n-1}(a)|, \sup_{s\in I} |x_n(s)|];$$

where subscripts indicate derivatives and a is an arbitrary fixed element of I. The double and triple norms $||x||$ and $|||x|||$ are equivalent: either one is majorized by a constant times the other, as is clear from the Taylor formulas for $x_i(s)$, $s\in I$, $i<n$, in terms of $x(a), \cdots, x_{n-1}(a)$, and $x_n(s)$, $s\in I$.

A functional $F\in C_n(I)^*$ has norms $||F||$ and $|||F|||$ relative to the double and triple norm in $C_n(I)$, respectively. The norms $||F||$ and $|||F|||$ are equivalent. One advantage of $||F||$ is that it is given explicitly in the next theorem, for an arbitrary $F\in C_n(I)^*$, whereas the calculation of $|||F|||$ may be awkward.

If f is a function of bounded variation on I, we agree to extend its definition as follows:

$$f(s) = \begin{cases} f(\alpha) & \text{if } s \leqq \alpha, \\ f(\tilde{\alpha}) & \text{if } s \geqq \tilde{\alpha}, \end{cases}$$

where $I=\{s: \alpha\leqq s\leqq \tilde{\alpha}\}$. We say that f is a *normalized* function of bounded variation if f is of bounded variation and $f(\alpha)=0$, $f(s+0) =f(s)$ whenever $s\neq\alpha$. Thus a normalized function of bounded variation on I vanishes on the lower boundary of I and is continuous from above except possibly on the lower boundary.

THEOREM 1. *Suppose that $F\in C_n(I)^*$. Take $a\in I$. Then unique constants $c^0, c^1, \cdots, c^{n-1}$ and a unique normalized function λ of bounded variation exist such that*

$$Fx = \sum_{i=0}^{n-1} c^i x_i(a) + \int_I x_n(s)d\lambda(s) \quad \textit{for all } x \in C_n(I).$$

Furthermore,

$$i!c^i = F[(s-a)^i],$$

$$\lambda(t) = \begin{cases} \lim\limits_{\nu=1,2,\cdots} FT_s^n\theta^\nu(t, s) & \textit{if } t > \alpha, \\ 0 \quad \textit{otherwise}; \end{cases}$$

and

$$\|F\| = \sum_{i=0}^{n-1} |c^i| + \operatorname{var} \lambda.$$

Here the i attached to $(s-a)$ is an exponent; T_s is the Taylor operator of taking the indefinite integral which vanishes at $s=a$:

$$T_s z(s) = \int_a^s z(\bar{s})\, d\bar{s};$$

T_s^n is the n-fold iteration of T_s, which may be expressed as a single integral [5, p. 152]; $\{\theta^\nu : \nu = 1, 2, \cdots\}$ is a monotone sequence of continuous functions whose limit is the Heaviside function θ:

$$\theta(t, s) = \begin{cases} 0 & \text{if } t < s, \\ 1 & \text{if } s \leqq t, \end{cases} \tag{1}$$

and var λ is the variation of λ. In the equation for $\lambda(t)$, F operates on its argument as a function of s. The theorem asserts that the limit in the above definition of λ exists.

If $n=0$, Theorem 1 reduces to Riesz's theorem on $C_0(I)^*$. If $n>0$, Theorem 1 is an immediate consequence of Riesz's theorem. All details are given in [5, pp. 139, 154].

3. **The space $C_n^m(D)$.** There are many generalizations of $C_n(I)$. One is the space $C_n^m(D)$ defined as follows. Let D be a subset of Euclidean m-space R^m. A function x on D to R is an element of $C_n^m(D)$ if and only if there exists a function y on an open set $\Omega \supset D$ which is an extension of x and which has continuous nth partial derivatives on Ω. The open set Ω may depend on x. The partial derivatives of x are defined as those of one such extension y [6].

We define the triple norm $|||x|||$ in $C_n^m(D)$ as

$$|||x||| = \max_{\sigma(h) \leqq n} \sup_{(s) \in D} |x_{(h)}(s)|, \qquad x \in C_n^m(D),$$

where

$$(s) = (s_1, \cdots, s_m), \quad (h) = (h_1, \cdots, h_m), \quad \sigma(h) = h_1 + \cdots + h_m.$$

The indices $h_1, \cdots, h_m$ are nonnegative integers, and the compound subscript (h) indicates a partial derivative.

If D is compact, which we shall always assume, then $|||x|||$ is finite whenever $x \in C_n^m(D)$.

Let us say that a set D is *boundedly connected* if any two points of D may be joined by a rectifiable curve contained in D, of uniformly bounded length.

Suppose that D is compact and boundedly connected, and that

(a) is a fixed element of D. We define the double norm $\|x\|$ in $C_n^m(D)$ as

$$\|x\| = \max_{\sigma(h)<n;\,\sigma(j)=n} \left[\left| x_{(h)}(a) \right|, \sup_{(s)\in D} \left| x_{(j)}(s) \right| \right], \qquad x \in C_n^m(D).$$

Then $\|x\|$ is majorized by $|||x|||$ and, conversely, $|||x|||$ is majorized by a constant times $\|x\|$, since we may express $x_{(h)}(s)$, $(s)\in D$, $\sigma(h)<n$, in terms of $x_{(h)}(a)$, $\sigma(h)<n$, and $x_{(j)}(s)$, $(s)\in D$, $\sigma(j)=n$, by Whitney's form of Taylor's formula along rectifiable curves of bounded length [6, equation (4)]. Thus the double and triple norms in $C_n^m(D)$ are equivalent if D is boundedly connected.

THEOREM 2. *Suppose that $F\in C_n^m(D)^*$, where $D\subset \mathrm{R}^m$ is compact and boundedly connected. Take $(a)\in D$. Put*

$$c^{(h)} = F[(s_1 - a_1)^{h_1} \cdots (s_m - a_m)^{h_m}]/h_1! \cdots h_m!, \qquad \sigma(h) < n.$$

Then functions $f^{(j)}$, $\sigma(j)=n$, of bounded variation on D, exist such that

$$Fx = \sum_{\sigma(h)<n} c^{(h)} x_{(h)}(a) + \sum_{\sigma(j)=n} \int_D x_{(j)}(s)\, df^{(j)}(s) \qquad \text{for all } x \in C_n^m(D), \tag{2}$$

and

$$\|F\| = \sum_{\sigma(h)<n} \left| c^{(h)} \right| + \sum_{\sigma(j)=n} \operatorname{var} f^{(j)}. \tag{3}$$

The functions $f^{(j)}$ may, alternatively, be called bounded signed measures.

A few comments before the proof may be of interest.

Theorem 2 does not afford a method of calculating the functions $f^{(j)}$, $\sigma(j)=n$. Nor is any universal method known, even in the case in which D is a solid sphere or an m-dimensional interval! In considering a particular functional F, one may, perhaps, find functions $f^{(j)}$ for which (2) holds; then (2) would imply that [5, p. 204]

$$\|F\| \leq \sum_{\sigma(h)<n} \left| c^{(h)} \right| + \sum_{\sigma(j)=n} \operatorname{var} f^{(j)}, \tag{4}$$

a relation which is weaker than (3). The reason that equality in (4) may not be valid is that the partial derivatives $x_{(j)}$, $\sigma(j)=n$, of a function $x\in C_n^m(D)$ are somewhat dependent on one another; if $x_{(j_0)}$ resonates with its integrator $df^{(j_0)}$ in (2), it may be impossible for $x_{(k)}$ to resonate with $df^{(k)}$, $\sigma(k)=n$. Thus the full resonance indicated by (3) instead of (4) may be unattainable and unapproachable for $x\in C_n^m(D)$, $\|x\|=1$. For the functions $f^{(j)}$ of Theorem 2, however, both (2) and (3) hold.

The general case is like the particular case $m=2$, $n=1$, which we now discuss, using an alphabetical notation:

$$(a, b) \in D \subset \mathbf{R}^2,$$

$$|||x||| = \max[\sup|x(s,t)|, \sup|x_{1,0}(s,t)|, \sup|x_{0,1}(s,t)|],$$

$$\|x\| = \max[|x(a,b)|, \sup|x_{1,0}(s,t)|, \sup|x_{0,1}(s,t)|], \qquad x \in C_1^2(D);$$

where the suprema are taken for $(s, t) \in D$.

A first attempt to prove Theorem 2 might start with Taylor's formula,

$$x(s,t) = x(a,b) + \int_0^1 \{(s-a)x_{1,0}[a+u(s-a), b+u(t-b)] + (t-b)x_{0,1}[a+u(s-a), b+u(t-b)]\}\,du, \quad (s,t) \in D,$$

valid for $x \in C_1^2(D)$, where, for the moment, we assume that D is convex. If $F \in C_1^2(D)^*$, we may operate with F on both sides of the equation, but F of the integral is not readily simplified. One may not interchange F and $\int$, since the integrand is not necessarily an element of $C_1^2(D)$ for fixed u. Nor may we write F of the integral as the sum of two terms of which one is

$$F\int_0^1 (s-a)x_{1,0}[a+u(s-a), b+u(t-b)]\,du,$$

since the argument of F here is not necessarily an element of $C_1^2(D)$.

PROOF OF THEOREM 2. The particular case $m=2$, $n=1$, will indicate the general proof. Let

$$Y = \mathbf{R} \times C_0^2 \times C_0^2 = \{(\gamma, y, z) : \gamma \in \mathbf{R},\ y \in C_0^2, \text{ and } z \in C_0^2\},$$

with

$$\|(\gamma, y, z)\|_Y = \max(|\gamma|, \|y\|_{C_0^2}, \|z\|_{C_0^2}),$$

where $C_0^2 = C_0^2(D)$. The key to the present proof is that if $x \in C_1^2(D)$, then $(x(a,b), x_{1,0}, x_{0,1}) \in Y$.

Let M be the linear set

$$\{(\gamma, y, z) : \text{For some } x \in C_1^2(D), \gamma = x(a,b), y = x_{1,0}, \text{ and } z = x_{0,1}\} \subset Y.$$

Define ϕ as the map of $C_1^2(D)$ onto M in which

$$\phi(x) = (x(a,b), x_{1,0}, x_{0,1}) \in M, \qquad x \in C_1^2(D).$$

By Whitney's form of Taylor's formula and our hypothesis on D,

ϕ is one-to-one. Furthermore, both ϕ and ϕ^{-1} are bounded maps with bound 1, since

$$\|\phi(x)\|_Y = \|x\|_{C^2_1(D)}, \qquad x \in C^2_1(D).$$

Put

$$G = F\phi^{-1}.$$

Thus G is a linear functional on $M \subset Y$, and G is bounded with

$$\|G\|_{M^*} = \|F\|_{C^2_1(D)^*} < \infty.$$

By the Hahn-Banach theorem [1, p. 55], there exists a linear continuous functional H on Y such that

$$H(\gamma, y, z) = G(\gamma, y, z) \quad \text{for all } (\gamma, y, z) \in M$$

and

$$\|H\|_{Y^*} = \|G\|_{M^*}.$$

Now

$$H(\gamma, y, z) = H(\gamma, 0, 0) + H(0, y, 0) + H(0, 0, z),$$

and the terms on the right are linear continuous functionals on $\mathbf{R}$, C^2_0, C^2_0, respectively. Hence

$$H(\gamma, y, z) = c\gamma + \int\int_D y(s, t)\, de(s, t) + \int\int_D z(s, t)\, df(s, t), \qquad (\gamma, y, z) \in Y,$$

and

$$\|H\|_{Y^*} = |c| + \operatorname{var} e + \operatorname{var} f,$$

where $c = H(1, 0, 0) = F[1] \in \mathbf{R}$, and e, f are functions of bounded variation on D for which explicit formulas in terms of H can be given [5, pp. 244, 245].

Then

$$Fx = G\phi(x) = H\phi(x) = H[x(a, b), x_{1,0}, x_{0,1}]$$
$$= cx(a, b) + \int\int_D x_{1,0}(s, t)\, de(s, t) + \text{dual term}, \qquad x \in C^2_1(D).$$

This completes the proof.

In a similar fashion, one may establish the following theorem.

THEOREM 2'. *Suppose that $F \in C^m_n(D)^*$, where $D \subset \mathbf{R}^m$ is compact but* *not necessarily connected. Then functions $g^{(h)}$, $\sigma(h)\leqq n$, of bounded variation on D, exist such that*

$$Fx = \sum_{\sigma(h)\leqq n} \int_D x_{(h)}(s)\, dg^{(h)}(s) \quad \textit{for all } x \in C_n^m(D),$$

and

$$|||F||| = \sum_{\sigma(h)\leqq n} \operatorname{var} g^{(h)}.$$

Here, too, there is no known method of finding the functions $g^{(h)}$, $\sigma(h)\leqq n$. Theorem 2′, with $m=1$, is a partial analogue of Theorem 1.

An interesting question is this: When can an expression Fx of the form (2) vanish for all $x\in C_n^m(D)$? By taking $x(s)$ to be the polynomial $(s_1-a_1)^{h_1}\cdots(s_m-a_m)^{h_m}$, we see at once that it is necessary that $c^{(h)}=0$ for all (h) such that $\sigma(h)<n$.

Let I be a compact interval which contains D. If $df^{(j)}$, $\sigma(j)=n$, are given on D, then $df^{(j)}$ may be extended onto I by ascribing zero measure to all subsets of $I-D$. Then

$$\int_D z\, df^{(j)} = \int_I z\, df^{(j)},$$

whenever the first integral exists. We shall, therefore, consider expressions of the form (2) in which D is a compact interval of R^m and $c^{(h)}=0$, $\sigma(h)<n$.

Let

$$I = \{(s) : \alpha_1 \leqq s_1 \leqq \tilde{\alpha}_1, \cdots, \alpha_m \leqq s_m \leqq \tilde{\alpha}_m\} \subset \mathsf{R}^m.$$

If f is a function of bounded variation on I, we agree to extend its definition as follows:

$$f(s) = f(s') \quad \text{for all } (s) \in \mathsf{R}^m,$$

where

$$s_i' = \begin{cases} \alpha_i & \text{if } s_i \leqq \alpha_i, \\ s_i & \text{if } \alpha_i \leqq s_i \leqq \tilde{\alpha}_i, \\ \tilde{\alpha}_i & \text{if } \tilde{\alpha}_i \leqq s_i, \end{cases} \qquad i = 1, \cdots, m.$$

We say that f is a *normalized* function of bounded variation on I if f vanishes on the lower boundary of I and, except possibly on the lower boundary, is continuous from above: $f(s)=0$ if for some i, $s_i=\alpha_i$, and $f(s+0)=f(s)$ if for all i, $s_i\neq\alpha_i$.

In the following theorem, the operator $D_i=\partial/\partial s_i$ indicates partial differentiation, $i=1,\cdots,m$; the operator S_i indicates the substitution of $\tilde{\alpha}_i$ for s_i; the operator $T_i = T_{s_i}$ is the analogue of the Taylor operator of Theorem 1; and a caret above an operator indicates its absence. For example,

$$\hat{S}_1 S_2 \hat{S}_3 T_1 T_3 z(s_1, s_2, s_3) = \int_{\alpha_1}^{s_1} d\bar{s}_1 \int_{\alpha_3}^{s_3} z(\bar{s}_1, \tilde{\alpha}_2, \bar{s}_3)\, d\bar{s}_3.$$

THEOREM 3. *Suppose that g^i, $i = 1, \cdots, m$, are normalized functions of bounded variation on I. A necessary and sufficient condition that*

$$\sum_{i=1}^{m} \int_I (D_i x)\, dg^i = 0 \quad \textit{for all } x \in C_1^m(I)$$

is that the following conditions hold for all $(s) \in I$:

$$S_1 \cdots \hat{S}_i \cdots S_m g^i(s) = 0, \qquad i = 1, \cdots, m;$$

$$S_1 \cdots \hat{S}_i \cdots \hat{S}_j \cdots S_m[T_j g^i(s) + T_i g^j(s)] = 0, \ i < j;\ i, j = 1, \cdots, m;$$

$$S_1 \cdots \hat{S}_i \cdots \hat{S}_j \cdots \hat{S}_k \cdots S_m[T_j T_k g^i(s) + T_k T_i g^j(s) + T_i T_j g^k(s)] = 0,$$
$$i < j < k;\ i, j, k = 1, \cdots, m;$$
$$\vdots$$
$$\sum_{i=1}^{m} T_1 \cdots \hat{T}_i \cdots T_m g^i(s) = 0.$$

We shall give the proof for the case $m = 2$, where the theorem is the following.

Suppose that $I = I_s \times I_t$, $I_s = [\alpha, \tilde{\alpha}]$, $I_t = [\beta, \tilde{\beta}]$, and e, f are normalized functions of bounded variation on I. A necessary and sufficient condition that

$$(5) \quad \iint_I x_{1,0}(s, t)\, de(s, t) + \iint_I x_{0,1}(s, t)\, df(s, t) = 0 \quad \textit{for all } x \in C_1^2(I)$$

is that

$$(6) \quad \begin{aligned} e(s, \tilde{\beta}) &= 0 \quad \textit{for all } s \in I_s, \\ f(\tilde{\alpha}, t) &= 0 \quad \textit{for all } t \in I_t, \end{aligned}$$

and

$$(7) \quad \int_\beta^t e(s, \bar{t})\, d\bar{t} + \int_\alpha^s f(\bar{s}, t)\, d\bar{s} = 0 \quad \textit{for all } (s, t) \in I.$$

PROOF. Denote the left side of (5) by Fx. Suppose that $y \in C_1(I_s)$ and that $x(s, t) = y(s)$, $(s, t) \in I$. Then $x \in C_1^2(I)$, and [5, p. 518]

$$Fx = \iint_I y_1(s)\, de(s, t) = \int_{I_s} y_1(s)\, de(s, \tilde{\beta}).$$

This expression vanishes for all $y \in C_1(I_s)$ if and only if $e(s, \tilde{\beta}) = 0$, by Riesz's theorem [5, p. 135; cf. p. 507 also], since our hypothesis that $e(s, t)$ is a normalized function of bounded variation on I implies that $e(s, \tilde{\beta})$ is a normalized function of bounded variation on I_s.

Thus (6) is necessary and sufficient that $Fx = 0$ for all $x \in C_1^2(I)$ which are functions of s alone or t alone.

Assume (6). We shall show that $Fx = 0$ for all $x \in C_1^2(I)$ if and only if (7) holds. Since $C_2^2(I)$ is dense in $C_1^2(I)$, it will be sufficient to consider $C_2^2(I)$.

Consider an arbitrary $x \in C_2^2(I)$. By a simple Taylor expansion,

$$(8)\quad x(s, t) = x(\alpha, t) + \int_\alpha^s x_{1,0}(\bar{s}, \beta)\, d\bar{s} + \int_\alpha^s d\bar{s} \int_\beta^t x_{1,1}(\bar{s}, \bar{t})\, d\bar{t}, \quad (s, t) \in I.$$

This relation is, in fact, equation (56) of [5, p. 184] for the space B in which $(a, b) = (\alpha, \beta)$ and

$$\bar{\omega}_{s,t} = \{(1, 1)\}, \quad \bar{\omega}_{s,b} = \{(1, 0)\}, \quad \bar{\omega}_{a,t} = \{(0, 0)\}, \quad \bar{\omega}_{a,b} = 0.$$

Since the first two terms on the right of (8) are functions of s alone or t alone, they are zeros of F. Hence

$$\begin{aligned} Fx &= \iint_I de(s, t) \int_\beta^t x_{1,1}(s, \bar{t})\, d\bar{t} + \text{dual term} \\ &= \iint_I de(s, t) \int_{I_{\bar{t}}} x_{1,1}(s, \bar{t})\theta(t, \bar{t})\, d\bar{t} + \text{dual term}, \end{aligned}$$

where θ is the Heaviside function (1). By Fubini's theorem,

$$\begin{aligned} Fx &= \int_{I_{\bar{t}}} d\bar{t} \iint_{I_s \times I_t} x_{1,1}(s, \bar{t})\theta(t, \bar{t})\, de(s, t) + \text{dual} \\ &= \int_{I_{\bar{t}}} d\bar{t} \iint_{I_s \times [\bar{t}, \tilde{\beta}]} x_{1,1}(s, \bar{t})\, de(s, t) + \text{dual} \\ &= \int_{I_{\bar{t}}} d\bar{t} \int_{I_s} x_{1,1}(s, \bar{t})\, [de(s, \tilde{\beta}) - de(s, \bar{t} - 0)] + \text{dual} \\ &= -\int_{I_{\bar{t}}} d\bar{t} \int_{I_s} x_{1,1}(s, \bar{t})\, de(s, \bar{t}) + \text{dual}, \end{aligned}$$

by (6) and the fact that $e(s, t)$ and $e(s, t-0)$ differ on a countable set which is therefore of Lebesgue measure zero. Hence

$$Fx = -\int\int_I x_{1,1}(s,t)\, d_{s,t}\left[\int_\beta^t e(s,\bar t)\, d\bar t + \int_\alpha^s f(\bar s, t)\, d\bar s\right],$$

by a direct argument. Now the integrator (quantity in brackets) is a normalized function of bounded variation on I. Hence $Fx=0$ for all $x_{1,1}\in C_0(I)$ if and only if the integrator vanishes for all $(s, t)\in I$ [5, p. 244]. This establishes (7) and completes the proof.

We may construct many forms of $0\in C_1^2(I)^*$ as follows. Let Γ be an oriented rectifiable closed curve contained in I. Then

$$\int_\Gamma dx = \int_\Gamma x_{1,0}(s,t)\, ds + \int_\Gamma x_{0,1}(s,t)\, dt = 0 \quad \text{for all } x \in C_1^2(I).$$

Now express the integral on each partial as a double Stieltjes integral; for example,

$$\int_\Gamma x_{1,0}(s,t)\, ds = \int\int_I x_{1,0}(s,t)\, de(s,t),$$

where e is the normalization [5, p. 532] of the function η defined as follows: $\eta(\bar s, \bar t)$ equals the difference in the s-coordinates of the last point of Γ in $[\alpha, \bar s]\times[\beta, \bar t]$ and the first point therein. With the dual definition of f, we now have an instance of (5).

Theorem 3 generalizes to $C_n^m(I)$ but both statement and proof become complicated. Perhaps it will be suitable to consider only $C_2^2(I)$.

THEOREM 4. *Suppose that e, f, g are normalized functions of bounded variation on I. A necessary and sufficient condition that*

$$\int\int_I x_{2,0}(s,t)\, de(s,t) + \int\int_I x_{1,1}(s,t)\, df(s,t) + \int\int_I x_{0,2}(s,t)\, dg(s,t) = 0 \quad \textit{for all } x \in C_2^2(I) \tag{9}$$

is that

$$f(\tilde\alpha, \tilde\beta) = 0, \tag{10}$$

$$e(s, \tilde\beta) = 0 \quad \textit{for all } s \in I_s, \qquad g(\tilde\alpha, t) = 0 \quad \textit{for all } t \in I_t, \tag{11}$$

$$\begin{aligned} T_s f(s,\tilde\beta) + \int_\beta^{\tilde\beta} e(s,\bar t)\, d\bar t &= 0 \quad \textit{for all } s \in I_s, \\ T_t f(\tilde\alpha, t) + \int_\alpha^{\tilde\alpha} g(\bar s, t)\, d\bar s &= 0 \quad \textit{for all } t \in I_t, \end{aligned} \tag{12}$$

and

(13) $\quad T_t^2 e(s, t) + T_s T_t f(s, t) + T_s^2 g(s, t) = 0 \quad$ *for all* $(s, t) \in I$.

PROOF. Denote the left side of (9) by Fx. Suppose that $y \in C_2(I_s)$ and that

$$x(s, t) = y(s), \qquad (s, t) \in I.$$

Then $x \in C_2^2(I)$, and

$$Fx = \int\int_I y_2(s)\, de(s, t) = \int_{I_s} y_2(s)\, de(s, \tilde{\beta}).$$

This expression vanishes for all $y \in C_2(I_s)$ if and only if $e(s, \tilde{\beta}) = 0$, since $e(s, \tilde{\beta})$ is normalized on I_s. Thus (11) is necessary and sufficient that $Fx = 0$ for all $x \in C_2^2(I)$ which are functions of s alone or of t alone.

Assume (11). Put $x(s, t) = st$. Then

$$Fx = \int\int_I df(s, t) = f(\tilde{\alpha}, \tilde{\beta}) = 0$$

if and only if (10) holds. Assume (10). Suppose that $y \in C_2(I_s)$ and that $x(s, t) = (t - \beta)y(s)$, $(s, t) \in I$. Then

$$\begin{aligned} Fx &= \int\int_I (t - \beta) y_2(s)\, de(s, t) + \int\int_I y_1(s)\, df(s, t) \\ &= \int_{I_s} y_2(s)\, d_s \int_{I_t} (t - \beta)\, d_t e(s, t) + \int_{I_s} y_1(s)\, df(s, \tilde{\beta}), \end{aligned}$$

by the dual of Fubini's theorem, given in the appendix of the present paper. By parts, using (10) and (11), we see that

$$\begin{aligned} Fx &= \int_{I_s} y_2(s)\, d_s\left[0 - \int_{I_t} e(s, t)\, dt\right] + 0 - \int_{I_s} f(s, \tilde{\beta}) y_2(s)\, ds \\ &= -\int_{I_s} y_2(s)\, d_s\left[\int_{I_t} e(s, t)\, dt + \int_\alpha^s f(\bar{s}, \tilde{\beta})\, d\bar{s}\right] \\ &= -\int_{I_s} y_2(s)\, d_s\left[\int_{I_t} e(s, t)\, dt + T_s f(s, \tilde{\beta})\right]. \end{aligned}$$

Now the integrator is normalized on I_s. Hence $Fx = 0$ for all $y \in C_2(I_s)$ if and only if the first relation of (12) holds. This and the dual argument show that $Fx = 0$ for all $x \in C_2^2(I)$ which are such that either $x(s, t) = (t - \beta)y(s)$, $y \in C_2(I_s)$ or x is the dual function, if and only if (10) and (12) hold.

Assume (10), (11), and (12). We shall show that $Fx = 0$ for all

$x \in C_2^2(I)$ if and only if (13) holds. Since $C_4^2(I)$ is dense in $C_2^2(I)$, it will be sufficient to consider $C_4^2(I)$.

Consider an arbitrary $x \in C_4^2(I)$. By a simple Taylor expansion,

$$\begin{aligned} x(s, t) &= x(\alpha, t) + T_s x_{1,0}(s, \beta) + T_s T_t x_{1,1}(\alpha, t) + T_s^2 T_t x_{2,1}(s, \beta) \\ &\quad + T_s^2 T_t^2 x_{2,2}(s, t), \qquad (s, t) \in I. \end{aligned} \tag{14}$$

This relation is, in fact, equation (56) [5, p. 184] for the space B in which $(a, b) = (\alpha, \beta)$ and

$$\bar{\omega}_{s,t} = \{(2, 2)\}, \qquad \bar{\omega}_{s,b} = \{(1, 0), (2, 1)\},$$
$$\bar{\omega}_{a,t} = \{(0, 0), (1, 1)\}, \qquad \bar{\omega}_{a,b} = 0.$$

Now the terms on the right side of (14), except the last, are zeros of F. For example, $T_s^2 T_t x_{2,1}(s, \beta) = (t - \beta) T_s^2 x_{2,1}(s, \beta)$ and $T_s^2 x_{2,1}(s, \beta) \in C_2(I_s)$. Hence

$$\begin{aligned} Fx &= F T_s^2 T_t^2 x_{2,2}(s, t) \\ &= \iint_I [T_t^2 x_{2,2}(s, t)]\, de(s, t) + \iint_I [T_s T_t x_{2,2}(s, t)]\, df(s, t) \\ &\quad + \text{dual of first term} \\ &= \iint_I de(s, t) \int_{I_{\bar{t}}} x_{2,2}(s, \bar{t})(t - \bar{t})\theta(t, \bar{t})\, d\bar{t} \\ &\quad + \iint_I df(s, t) \iint_I x_{2,2}(\bar{s}, \bar{t})\theta(s, \bar{s})\theta(t, \bar{t})\, d\bar{s} d\bar{t} + \text{dual of first term}, \end{aligned}$$

by (1) and [5, p. 152]. By Fubini's theorem,

$$\begin{aligned} Fx &= \int_{I_{\bar{t}}} d\bar{t} \iint_I x_{2,2}(s, \bar{t})(t - \bar{t})\theta(t, \bar{t})\, de(s, t) \\ &\quad + \iint_I x_{2,2}(\bar{s}, \bar{t})\, d\bar{s} d\bar{t} \iint_I \theta(s, \bar{s})\theta(t, \bar{t})\, df(s, t) + \text{dual of first term}. \end{aligned}$$

Now, by the dual of Fubini's theorem,

$$\begin{aligned} &\iint_I x_{2,2}(s, \bar{t})(t - \bar{t})\theta(t, \bar{t})\, de(s, t) \\ &\qquad = \int_{I_s} x_{2,2}(s, \bar{t})\, d_s \int_{I_t} (t - \bar{t})\theta(t, \bar{t})\, d_t e(s, t); \end{aligned}$$

and, by (11),

$$\int_{I_t} (t-\bar{t})\theta(t,\bar{t})\, d_t e(s,t) = \int_{[\bar{t},\tilde{\beta}]} (t-\bar{t})\, d_t e(s,t) = 0 - \int_{\bar{t}}^{\tilde{\beta}} e(s,t)\, dt.$$

Also, by (10),

$$\int\int_I \theta(s,\bar{s})\theta(t,\bar{t})\, df(s,t)$$

$$= \int\int_{[\bar{s},\tilde{\alpha}]\times[\bar{t},\tilde{\beta}]} df(s,t) = -f(\tilde{\alpha},\bar{t}-0) - f(\bar{s}-0,\tilde{\beta}) + f(\bar{s}-0,\bar{t}-0);$$

and the last expression equals $-f(\tilde{\alpha},\bar{t}) - f(\bar{s},\tilde{\beta}) + f(\bar{s},\bar{t})$ except for countably many values of $\bar{s}$ and $\bar{t}$ [5, p. 524]. Since we may change the integrand of a Lebesgue integral on a set of measure zero,

$$\begin{aligned} Fx = &\int_{I_{\bar{t}}} d\bar{t} \int_{I_s} x_{2,2}(s,\bar{t})\, d_s \int_{\bar{t}}^{\tilde{\beta}} -e(s,t)\, dt \\ &+ \int\int_I x_{2,2}(\bar{s},\bar{t})[f(\bar{s},\bar{t}) - f(\bar{s},\tilde{\beta}) - f(\tilde{\alpha},\bar{t})]d\bar{s}d\bar{t} + \text{dual of first term} \\ = &\int\int_I x_{2,2}(s,t)\, d_{s,t}\left[\int_\beta^t dt^* \int_{t^*}^{\tilde{\beta}} -e(s,\bar{t})\, d\bar{t}\right. \\ &\left. + \int_\alpha^s \int_\beta^t [f(\bar{s},\bar{t}) - f(\bar{s},\tilde{\beta}) - f(\tilde{\alpha},\bar{t})]\, d\bar{s}d\bar{t} + \text{dual of first term}\right], \end{aligned}$$

by a direct argument. Hence, by (12),

$$\begin{aligned} Fx = &\int\int_I x_{2,2}(s,t) d_{s,t}[T_t(T_s f(s,\tilde{\beta}) + T_t e(s,t)) \\ &\qquad + T_s T_t(f(s,t) - f(s,\tilde{\beta}) - f(\tilde{\alpha},t)) + \text{dual of first term}] \\ = &\int\int_I x_{2,2}(s,t)\, d_{s,t}[T_t^2 e(s,t) + T_s T_t f(s,t) + T_s^2 g(s,t)]. \end{aligned}$$

It follows that $Fx=0$ for all $x\in C_4^2(I)$ if and only if (13) holds.

Thus Theorem 4 is established.

4. **The spaces** B, K, Z. Our knowledge of the adjoint X^* varies with the space X. We have seen that if $X=C_n^m(I)$, standard forms of $F\in C_n^m(I)^*$ are not accessible to us, if $n>0$ and $m>1$. It is therefore of interest to discover spaces X for which standard forms of $F\in X^*$ and of $\|F\|$ are known and utilizable. The spaces B, K, Z, to be described, are of this sort; B is a generalization of $C_n^1(I)$ and K of $C_{n-1}^1(I)$; Z is a subset of $C_0^m(I)$.

There are infinitely many spaces B, K [5, Chapters 6, 7]. I shall describe one pair of spaces in which, in the notation of the reference,

$$m = 2, \qquad p = 1, \qquad q = 2, \qquad n = p + q = 3.$$

Let $I = I_s \times I_t$ be a compact interval of the (s, t)-plane. Let $(a, b) \in I$. To define the space B, we first define the *core* of a function x on I as the set consisting of the following partial derivatives:

$$D_t D_s D_t x = x_{1,2}(s, t), \quad (s, t) \in I,$$

$$x_{2,0}(s, b), \quad D_s^2 D_t x \big|_{(s,t)=(s,b)} = x_{2,1}(s, b), \quad s \in I_s,$$

and

$$x_{0,4}(a, t), \qquad t \in I_t.$$

The *space* B is defined as the set of functions x for which the derivatives in the core exist and are continuous on I, I_s, I_t, respectively. We denote by $\omega_{s,t}$ the set consisting of the sole element $x_{1,2}(s, t)$, by $\omega_{s,b}$ the set consisting of the two elements $x_{2,0}(s, b)$ and $x_{2,1}(s, b)$, by $\omega_{a,t}$ the set consisting of the sole element $x_{0,4}(a, t)$. The core of x is $\omega_{s,t} \cup \omega_{s,b} \cup \omega_{a,t}$. We denote by $\omega_{a,b}$ the set of derivatives which are predecessors of derivatives in the core, each evaluated at (a, b). Thus $\omega_{a,b}$ is the set of six elements

$$x(a, b),\ x_{1,0}(a, b),\ x_{0,1}(a, b),$$
$$D_s D_t x \big|_{(s,t)=(a,b)} = x_{1,1}(a, b),\ x_{0,2}(a, b),\ x_{0,3}(a, b).$$

The *complete core* is defined as

$$\omega = \omega_{s,t} \cup \omega_{s,b} \cup \omega_{a,t} \cup \omega_{a,b}.$$

If $x \in B$, the elements of ω are determined uniquely. Conversely, we may take any ordered set of six constants as $\omega_{a,b}$, any ordered pair of continuous functions on I_s as $\omega_{s,b}$, the dual as $\omega_{a,t}$, and any continuous function on I as $\omega_{s,t}$; there is then a unique element x of B whose complete core ω is the constructed set. Thus ω is a set of coordinates (in fact, intrinsic coordinates) for x.

An order of differentiation has been specified for each element of ω. If $x \in B$, then certain derivatives of x must exist and be continuous. The set ϕ of these derivatives is called the *full core* of x. A straightforward elementary calculation shows that [5, p. 189]

$$\phi = \phi_{s,t} \cup \phi_{s,b} \cup \phi_{a,t},$$

where

$$\phi_{s,t} = \{x(s, t),\ x_{1,0}(s, t),\ x_{0,1}(s, t),\ x_{1,1}(s, t),\ x_{0,2}(s, t),\ x_{1,2}(s, t);\ (s, t) \in I;$$
$$\text{all orders of differentiation are allowed and equivalent}\};$$

$$\phi_{s,b} = \{x_{2,0}(s, b),\ x_{2,1}(s, b) = D_s x_{1,1}|_{(s,t)=(s,b)};\ s \in I_s;\ \text{both orders of differentiation in } x_{1,1} \text{ are allowed and equivalent}\};$$

$$\phi_{a,t} = \{x_{0,3}(a, t),\ x_{0,4}(a, t);\ t \in I_t\}.$$

In the present case only one order of differentiation in a mixed derivative is excluded: $x_{2,1}(s, b)$ may not be interpreted as $D_t x_{2,0}(s, t)\,|_{(s,t)=(s,b)}$.

We introduce two norms in B as follows: $\|x\|$ is the maximum of the suprema of the absolute values on I of the elements of ω, and $|||x|||$ is the analogous maximum for ϕ, where $x\in B$. These norms are equivalent. If a functional $F\in B^*$, its double norm $\|F\|$ is defined in terms of $\|x\|$.

THEOREM 5. *Suppose that $F\in B^*$. Then unique constants $c^{i,j}$ and normalized functions $\lambda^{i,j}$ of bounded variation on I_s, I_t, I, respectively, exist such that*

$$\begin{aligned} Fx = {} & \sum_{\omega_{a,b}} c^{i,j}x_{i,j}(a, b) + \sum_{\omega_{a,b}} \int_{I_s} x_{i,j}(s, b)\, d\lambda^{i,j}(s) + \textit{dual sum} \\ & + \int\!\!\int_I x_{p,q}(s, t)\, d\lambda^{p,q}(s, t) \quad \textit{for all } x \in B. \end{aligned} \tag{15}$$

Furthermore,

$$i!j!c^{i,j} = F[(s - a)^i(t - b)^j],$$

$$j!\lambda^{i,j}(\bar{s}) = \begin{cases} \lim_{\nu} F[(t - b)^j T_s^i \theta^\nu(\bar{s}, s)] & \textit{if } \bar{s} > \alpha, \\ 0 & \textit{otherwise}, \end{cases}$$

$$i!\lambda^{i,j}(\bar{t}) = \textit{dual expression},$$

$$\lambda^{p,q}(\bar{s}, \bar{t}) = \begin{cases} \lim_{\nu,\nu'} F[T_s^p T_t^q \theta^\nu(\bar{s}, s)\theta^{\nu'}(\bar{t}, t)] & \textit{if } \bar{s} > \alpha \textit{ and } \bar{t} > \beta, \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$\|F\| = \sum |c^{i,j}| + \sum \int_{I_s} d|\lambda^{i,j}|(s) + \textit{dual} + \int\!\!\int_I d|\lambda^{p,q}|(s, t).$$

The indices i, j here vary over the domains appropriate to the terms of (15) in which they appear.

This theorem, like Theorem 1, is an immediate consequence of Riesz's theorem on C_0^{m*}. The proof is given in [5, p. 246].

The formula (15) for Fx, $x\in B$, cannot be simplified, since the elements of ω are entirely independent of one another. The formula leads to many strong appraisals, of which

$$|Fx| \leqq \|F\| \|x\|, \qquad x \in B,$$

is one [5, p. 22].

If the functions $\lambda^{i,j}$ in (15) are absolutely continuous, the Stieltjes integrals reduce to ordinary integrals. Then the formula (15) is particularly useful: it permits appraisals by Hölder inequalities [5, p. 203] as well as exact evaluation by ordinary integrations. One may, for any $F\in B^*$, compute the functions $\lambda^{i,j}$ and, by direct study, determine whether $\lambda^{i,j}$ are absolutely continuous and, if so, calculate their densities. Such a calculation may be long and even impracticable. It may contain an element of unnecessary calculation, since the operators T_s, T_t in Theorem 5 are integrations and each differentiation of $\lambda^{i,j}$, where possible, undoes the effect of one integration.

The space K, to be described, permits direct access to an equality like (15) in which all integrators are absolutely continuous, with known densities. The space K involves the *retracted core* ρ and the *covered core* ξ of a function on I. The determination of ρ and ξ is straightforward [5, pp. 195, 262]. In the present case,

$$\rho = \rho_{s,t} \cup \rho_{s,b} \cup \rho_{a,t} \cup \rho_{a,b},$$

where

$$\begin{aligned}
\rho_{a,b} &= \omega_{a,b},\\
\rho_{s,b} &= \{x_{1,0}(s, b) - x_{1,0}(a, b), x_{1,1}(s, b) - x_{1,1}(a, b)\}, \quad x_{1,1} = D_sD_tx,\\
\rho_{a,t} &= \{x_{0,3}(a, t) - x_{0,3}(a, b)\},\\
\rho_{s,t} &= \{x_{0,1}(s, t) - x_{0,1}(s, b) - x_{0,1}(a, t) + x_{0,1}(a, b)\},
\end{aligned}$$

and

$$\xi = \xi_{s,t} \cup \xi_{s,b} \cup \xi_{a,t},$$

where

$$\begin{aligned}
\xi_{s,t} &= \{x(s, t), x_{0,1}(s, t)\},\\
\xi_{s,b} &= \{x_{1,0}(s, b), x_{1,1}(s, b)\}, \qquad x_{1,1} = D_sD_tx,\\
\xi_{a,t} &= \{x_{0,2}(a, t), x_{0,3}(a, t)\}.
\end{aligned}$$

We define the *space* K as the set of functions x on I for which the elements of ρ exist and are continuous.

If $x\in K$, then the elements of ξ must exist and be continuous. We introduce two norms in K as follows: $\|x\|$ is the maximum of the suprema of the absolute values on I of the elements of ρ, and $|||x|||$ is the analogous maximum for ξ. These norms are equivalent. Note that $B \subset K$ and $B^* \supset K^*$.

THEOREM 6. *Suppose that $F \in K^*$. Then unique constants $c^{i,j}$ and normalized functions $\kappa^{i,j}$ of bounded variation on I_s, I_t, I, respectively, exist such that*

$$
\begin{aligned}
Fx = {} & \sum_{\omega_{a,b}} c^{i,j} x_{i,j}(a, b) + \sum_{\omega_{s,b}} \int_{I_s} x_{i,j}(s, b) \kappa^{i,j}(s)\, ds + \textit{dual sum} \\
& + \iint_I x_{p,q}(s, t) \kappa^{p,q}(s, t)\, ds dt \quad \textit{for all } x \in B.
\end{aligned}
\tag{16}
$$

Furthermore,

$$
\begin{aligned}
i!j!c^{i,j} &= F[(s-a)^i (t-b)^j], \\
j!\kappa^{i,j}(\bar{s}) &= \lim_{\nu} F[(t-b)^j T_s^{i-1} \psi^{\nu}(a, \bar{s}, s)] \quad \textit{if } \bar{s} > \alpha, \\
i!\kappa^{k,j}(\bar{t}) &= \textit{dual expression}, \\
\kappa^{p,q}(\bar{s}, \bar{t}) &= \lim_{\nu,\nu'} F[T_s^{p-1} T_t^{q-1} \psi^{\nu}(a, \bar{s}, s) \psi^{\nu'}(b, \bar{t}, t)] \quad \textit{if } \bar{s} > \alpha \textit{ and } \bar{t} > \beta.
\end{aligned}
$$

Here, $\psi^{\nu}(a, \bar{s}, s) = \theta^{\nu}(\bar{s}, a) - \theta^{\nu}(\bar{s}, s)$, $\nu = 1, 2, \cdots$, are a standard sequence of continuous functions [5, p. 146]. The proof of Theorem 6 is given in [5, pp. 266, 270].

It is Theorem 6 which justifies the study of the space K. Its hypothesis involves intrinsic properties of F. Thus $F \in K^*$ means that Fx is defined wherever $x \in K$, that F is linear on K, and that F is continuous on K. Of course, Theorems 1, 2, and 5 also involve intrinsic properties of their functionals. The earlier theorems, however, are immediate consequences of Riesz's theorem, whereas Theorem 6 is a somewhat removed consequence. The proof of Theorem 6 depends on the exact definition of K and its norm; this definition is just contrived to counter difficulties related to the partial dependence of partial derivatives of x. The hypothesis of Theorem 6 cannot be weakened.

An elementary application of Theorem 6 is the following. Let $F = R$ be the remainder

$$
Rx = \iint_I x(s, t)\, d\mu(s, t) - \gamma x(s^0, t^0)
$$

in the approximation of the double integral by the natural multiple γ of the integrand $x(s^0, t^0)$ at the center of mass, where μ is an arbitrary fixed function of bounded variation on I, and

$$\gamma = \iint_I d\mu(s, t), \quad \gamma s^0 = \iint_I s\, d\mu(s, t), \quad \gamma t^0 = \iint_I t\, d\mu(s, t).$$

We assume that $\gamma \neq 0$ and that $(s^0, t^0) \in I$. The functional R is defined for all functions which are μ-integrable and which are defined at (s^0, t^0). We shall consider restrictions of R, which we continue to denote by the same letter R. Then $R \in K^*$ for all spaces K. We have infinitely many formulas (16) for Rx, $x \in B$, one for each space B which has a companion K. Each formula is accessible; each gives Rx in terms of independent elements; each is sharply appraisable. The effect of our having used the center of mass and the factor γ is that

$$c^{0,0} = c^{1,0} = c^{0,1} = 0.$$

Whether other coefficients $c^{i,j}$ are present in (16) depends on $\omega_{a,b}$ and μ.

The proof of Theorem 6 involves another function space Z. As Z seems interesting in itself, I shall describe it. The *space* Z is defined as the subspace of $C_0^2(I)$ consisting of functions $x(s, t)$ on I which vanish everywhere on I_s when $t = b$ and on I_t when $s = a$:

$$Z = \{x \in C_0^2(I) : x(s, b) = 0 = x(a, t) \text{ for all } s \in I_s \text{ and } t \in I_t\},$$

with the same norm as in $C_0^2(I)$:

$$\|x\| = \sup_{(s,t)\in I} |x(s, t)|, \qquad x \in Z.$$

Consider a functional $F \in Z^*$. Since $Z \subset C_0^2(I)$, the Hahn-Banach theorem implies that there is an extension $G \in C_0^2(I)^*$ of F with the same norm, and Riesz's theorem gives an expression for Gx, $x \in C_0^2(I)$, as a Stieltjes integral on x. The next theorem gives an accessible and useful representation of F, different from the Hahn-Banach extension.

THEOREM 7. *Suppose that $F \in Z^*$. There is a unique normalized function λ of bounded variation on I which vanishes everywhere on the boundary of I such that*

$$Fx = \iint_I x(s, t)\, d\lambda(s, t) \quad \text{for all } x \in Z.$$

Furthermore,

$$\lambda(\bar{s}, \bar{t}) = \begin{cases} \lim_{\nu,\nu'} F[\psi^\nu(a, \bar{s}, s)\psi^{\nu'}(b, \bar{t}, t)] & \text{if } \bar{s} > \alpha \text{ and } \bar{t} > \beta, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\|F\| = \iint_{s\neq a;\, t\neq b} d|\lambda|(s, t).$$

The proof is given in [5, p. 257]. We may transform the integral for Fx by parts in a particularly simple fashion because λ vanishes everywhere on the boundary of I [5, p. 518].

5. **Factors of operators.** Let $\mathfrak{J}(X, Y)$ denote the space of linear continuous maps on X to Y, with norm

$$\|T\| = \sup_{x\in X;\, \|x\|=1} \|Fx\|, \qquad T \in \mathfrak{J}(X, Y),$$

where X and Y are normed linear spaces. The space $\mathfrak{J}(X, Y)$ is determined by X and Y. A description of much of our knowledge about $\mathfrak{J}(X, Y)$ for specific spaces X, Y is given in [2, Chapters 4, 6]. If Y is the number system, then $\mathfrak{J}(X, Y) = X^*$, the case considered heretofore.

If $T\in\mathfrak{J}(X, Y)$, this fact alone sometimes permits us to acquire an explicit expression for Tx, $x\in X$. The space $\mathfrak{J}(X, Y)$, however, may be so complicated that we have no practicable universal method for expressing T in standard useful form.

An analysis of an individual T into factors may be useful. If $T=QU$, where Q and U are linear operators, then $Tx=0$ whenever $Ux=0$, $x\in X$. Conversely, if $Tx=0$ whenever $Ux=0$, $x\in X$, where U is a linear continuous operator, we may ask whether a linear continuous operator Q exists such that $T=QU$.

THEOREM 8. *Suppose that X, $\tilde{X}$, Y are Banach spaces, that*

$$T \in \mathfrak{J}(X, Y), \qquad U \in \mathfrak{J}(X, \tilde{X}), \qquad \tilde{X} = UX.$$

If $Tx=0$ whenever $Ux=0$, $x\in X$, then there exists a unique linear continuous operator $Q\in\mathfrak{J}(\tilde{X}, Y)$ such that

$$Tx = QUx \quad \textit{for all } x \in X. \tag{17}$$

The proof is given in [5, p. 311].

That Q is continuous is an important part of the conclusion, for continuity of Q means that the factorization $T=QU$ involves no loss of smoothness. Continuity of Q implies the sharp appraisal

$$\|Tx\| \leqq \|Q\|\,\|Ux\|, \qquad x \in X,$$

where $\|Q\| < \infty$.

Theorem 8 depends on Banach's theorem of 1929 on the continuity of the inverse of a linear continuous operator [1, p. 41], [5, p. 307]. Completeness of the spaces enters.

Suppose that the hypothesis in Theorem 8 is lightened in that we do not require X, $\tilde{X}$, Y to be complete. We may then complete X and Y, and T and U. Thereafter put $\tilde{X} = UX$. Then the hypothesis of Theorem 8 will be in force except in one respect: the normed linear space $\tilde{X}$ may not be complete. Then the conclusion of Theorem 8 will be in force except in one respect: the linear operator Q will exist and be closed but perhaps not continuous.

A plan for the analysis of $T \in \mathfrak{J}(X, Y)$, where X and Y are Banach spaces, is as follows. Seek a linear continuous operator U on X to some normed linear space such that $Tx = 0$ whenever $Ux = 0$, $x \in X$. Then ascertain whether UX is complete.

In the past, U has often been taken as n-fold differentiation: $U = D^n$, when $X = C_n(I)$, $I \subset \mathbf{R}$. The condition $Ux = 0$ then means that x is a polynomial of degree $n-1$ on I. In other instances U may be a homogeneous differential operator of order n, as in the next theorem. Alternatively, U may be a homogeneous difference operator or a mixed differential and difference operator. Further instances are given in [5, pp. 314, 315].

THEOREM 9. *Consider $x \in C_n(I)$, $I = \{s : \alpha \leqq s \leqq \tilde{\alpha}\}$. In the approximation of $x(t)$, $t \in I$, by a solution of the differential equation*

$$y_n + a^1 y_{n-1} + \cdots + a^n y = 0, \qquad a^1, \cdots, a^n \in C_0(I),$$

according to the criterion of least squares relative to a nonnegative measure μ on I, the remainder is

$$(Rx)(t) = \int_I [x_n(s) + a^1(s)x_{n-1}(s) + \cdots + a^n(s)x(s)]\lambda(s, t)\, ds \tag{18}$$

for all $t \in I$,

where the kernel λ can be described explicitly in terms of μ and any set of n independent solutions of the differential equation.

A proof based on Theorem 8 and an explicit description of λ are given in [5, p. 321]. The equality (18) is an instance of (17) with

$$U = D^n + a^1 D^{n-1} + \cdots + a^{n-1} D + a^n.$$

Theorem 9 is due to Radon [3]; cf. Rémès [4] and Widder [7]. What I should like to note particularly is that the theory of Banach spaces may be used to obtain explicit expressions for remainders in approximation.

6. **Appendix.** Fubini's theorem is a powerful tool in the study of

$$\int_{\alpha}^{\tilde{\alpha}}\int_{\beta}^{\tilde{\beta}} x(s, t)\, df(s, t) = \int_{\alpha}^{\tilde{\alpha}}\int_{\beta}^{\tilde{\beta}} x(s, t)\, d_{s,t}f(s, t)$$

if the integrator factors, that is, if $d_{s,t}f(s, t) = dg(s)dh(t)$. Dually, one would expect to be able to evaluate the double integral by two single integrations if the integrand factors, that is, if $x(s, t) = y(s)z(t)$. This is indeed the case, at least under the hypothesis of the following theorem.

THEOREM 10. *Suppose that f is a function of bounded variation on I and that $y \in C(I_s)$, $z \in C(I_t)$, where $I = I_s \times I_t$, $I_s = [\alpha, \tilde{\alpha}]$, $I_t = [\beta, \tilde{\beta}]$. Then*

$$\int_{\alpha}^{\tilde{\alpha}}\int_{\beta}^{\tilde{\beta}} y(s)z(t)\, df(s, t) = \int_{\alpha}^{\tilde{\alpha}} y(s)\, d_s\left[\int_{\beta}^{\tilde{\beta}} z(t)\, d_tf(s, t)\right]. \tag{19}$$

PROOF. Page references will be to [5, Chapter 12].
Put

$$g(s) = \int_{\beta}^{\tilde{\beta}} z(t)\, d_tf(s, t);$$

g is well-defined, since $f(s, t)$ is of bounded variation on I_t for each fixed s [p. 525].

Consider a subdivision $\{(s^i, t^j)\}$, $i = 0, \cdots, m$; $j = 0, \cdots, n$; of I [p. 516]. Now

$$\begin{aligned}\Delta g(s^{i-1}) = g(s^i) - g(s^{i-1}) &= \int_{\beta}^{\tilde{\beta}} z(t)\, d_t[f(s^i, t) - f(s^{i-1}, t)] \\ &= \int_{\beta}^{\tilde{\beta}}\int_{s^{i-1}}^{s^i} z(t)\, d_{s,t}f(s, t),\end{aligned}$$

by [p. 518]. Hence

$$|\Delta g(s^{i-1})| \leqq M\int_{\beta}^{\tilde{\beta}}\int_{s^{i-1}}^{s^i} dv(s, t)$$

and

$$\sum_{i=1}^{m} |\Delta g(s^{i-1})| \leqq Mv(\tilde{\alpha}, \tilde{\beta}).$$

where v is the total variation [p. 527] of f and

$$M = \sup_{t \in I_t} |z(t)|.$$

Hence g is of bounded variation and the right side of (19),

$$\int_{\alpha}^{\tilde{\alpha}} y(s)\, dg(s),$$

exists. The left side of (19) exists.

Put

$$\begin{aligned}\sigma &= \sum_{i,j\geqq 1} y(s^i)z(t^j)[f(s^i, t^j) - f(s^{i-1}, t^j) - f(s^i, t^{j-1}) + f(s^{i-1}, t^{j-1})] \\ &= \sum_{i,j} y(s^i)z(t^j) \int_{s^{i-1}}^{s^i} \int_{t^{j-1}}^{t^j} d_{s,t}f(s, t)\end{aligned}$$

and

$$\tau = \sum_{i\geqq 1} y(s^i)[g(s^i) - g(s^{i-1})] = \sum_{i} y(s^i) \int_{\beta}^{\tilde{\beta}} \int_{s^{i-1}}^{s^i} z(t)\, d_{s,t}f(s, t).$$

We know that σ and τ approach the left and right sides of (19) as the norm of the subdivision approaches zero. It is therefore sufficient to show that $\sigma - \tau \to 0$. But

$$\sigma - \tau = \sum_{i,j} y(s^i) \int_{s^{i-1}}^{s^i} \int_{t^{j-1}}^{t^j} [z(t^j) - z(t)]\, d_{s,t}f(s, t)$$

and

$$|\sigma - \tau| \leqq \sup_{s\in I_s} |y(s)| \sup_{|t'-t|\leqq \text{norm}} |z(t') - z(t)|\, v(\tilde{\alpha}, \tilde{\beta}) \to 0$$

as the norm of the subdivision $\to 0$. This completes the proof.

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