MULTIPLIERS OF FOURIER TRANSFORM IN A HALF-SPACE[1]

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1. Let (x, y) denote points in R^n where $x=(x_1, \cdots, x_{n-1})$, $y=x_n$. Points of the dual space are denoted by (ξ, η). Let Y_+ be the characteristic function of the half space $R^n_+ = \{(x, y) \mid y \geqq 0\}$. Let $M(\xi, \eta)$ be an $m \times m$ matrix-valued function whose entries are homogeneous functions:

$$M_{ij}(\lambda\xi, \lambda\eta) = M_{ij}(\xi, \eta), \qquad \lambda > 0, 1 \leqq i, j \leqq m.$$

Assume further that $M(\xi, \eta)$ is continuous and nonsingular for $(\xi, \eta) \neq 0$. Consider the bounded operator $\boldsymbol{M}$ in the space $(L^2(R^n_+))^m$ (with the natural norm denoted by $\| \;\|$):

$$(1) \qquad \boldsymbol{M}u = Y_+\mathfrak{F}^{-1}[M(\xi, \eta)(\mathfrak{F}u)(\xi, \eta)], \qquad u \in (L^2(R^n_+))^m,$$

where $\mathfrak{F}$ $(\mathfrak{F}^{-1})$ denotes the direct (inverse) Fourier transform with respect to all variables. $\mathfrak{F}_y$ $(\mathfrak{F}_x)$ will denote the transform with respect to y or x alone. The one-dimensional operator $\boldsymbol{M}_\xi$ is similarly defined in $(L^2(R^1_+))^m$ with the multiplier $M(\xi, \eta)$, ξ fixed:

$$(2) \qquad \boldsymbol{M}_\xi v = Y_+\mathfrak{F}_y^{-1}[M(\xi, \eta)(\mathfrak{F}_y v)(\eta)].$$

Our main results in this note are the following lemma and theorem.

LEMMA. *The estimate*

$$(3) \qquad \|u\| \leqq C\|\boldsymbol{M}u\|, \qquad u \in (L^2(R^n_+))^m$$

holds if and only if for all $|\xi| = 1$ (*uniformly*)

$$(4) \qquad \|v\| \leqq C\|\boldsymbol{M}_\xi v\|, \qquad v \in (L^2(R^1_+))^m.$$

For the scalar case $(m=1)$, we have

THEOREM. *Let* $M(\xi, \eta)$ *be a homogeneous function continuous and nonvanishing for* $(\xi, \eta) \neq 0$. *Let*

$$(5) \qquad -\frac{1}{2\pi}\int_{-\infty}^{\infty} d_\eta \arg M(\xi, \eta) = k + \theta, \qquad k \text{ integer}, -1/2 < \theta \leqq 1/2.$$

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If $\theta \neq 1/2$, then M has a closed range and is injective if $k \geqq 0$, surjective if $k \leqq 0$.

REMARKS. (1) The a priori L^2-estimates for mixed elliptic problems can be reduced to the validity of (3) [2], [3].

(2) For $n=1$, $M=M(\eta)$ is determined by $M(1)$ and $M(-1)$. The operator M is then a singular integral operator (with Cauchy kernel) on a half-line. For this case it was shown ([4], cf. also [1], [2], [5]) that M is invertible if and only if the matrix $M(1)^{-1}M(-1)$ does not have real negative eigenvalues.

2. PROOF OF THE LEMMA. Assume first that (4) holds, and apply it to

$$M\left(\frac{\xi}{|\xi|}, \eta\right) \quad \text{and} \quad v(y) = (\mathfrak{F}_x u)\left(\xi, \frac{y}{|\xi|}\right).$$

We get

$$\begin{aligned}\int_0^\infty \left|(\mathfrak{F}_x u)\left(\xi, \frac{y}{|\xi|}\right)\right|^2 dy &\leqq C^2 \int_0^\infty \left|\mathfrak{F}_y^{-1} M\left(\frac{\xi}{|\xi|}, \eta\right)\mathfrak{F}_y\left[\mathfrak{F}_x u\left(\xi, \frac{y}{|\xi|}\right)\right]\right|^2 dy \\ &= C^2 \int_0^\infty \left|\mathfrak{F}_y^{-1}[M(\xi, |\xi|\eta)|\xi|(\mathfrak{F}u)(\xi, |\xi|\eta)]\right|^2 dy \\ &= C^2 \int_0^\infty \left|(\mathfrak{F}_y^{-1} M \mathfrak{F} u)\left(\xi, \frac{y}{|\xi|}\right)\right|^2 dy.\end{aligned}$$

After changing variables on both sides (put $\bar{y}=y/|\xi|$), cancelling $|\xi|$ and integrating with respect to ξ, we have

$$\int_{R^{n-1}} \int_0^\infty |(\mathfrak{F}_x u)(\xi, y)|^2 dy d\xi \leqq C^2 \int_{R^{n-1}} \int_0^\infty |(\mathfrak{F}_y^{-1} M \mathfrak{F} u)(\xi, y)|^2 dy d\xi.$$

Using Parseval's identity (for $\mathfrak{F}_x^{-1}$) we obtain (3).

Assume now that (4) does not hold for some $\bar{\xi}$. Then for any $\epsilon > 0$ there is a $v_\epsilon(y) \in (L^2(R_+^1))^m$ such that $\|v_\epsilon(y)\| = 1$ and $\|M_{\bar{\xi}} v_\epsilon(y)\| \leqq \epsilon/2$. It is easily seen that $\|M_\xi v_\epsilon\| \leqq \epsilon$ if $|\xi_R - \bar{\xi}_R| < \delta$ and $\delta = \delta(\epsilon)$ is sufficiently small. Let now $w(\xi)$ be the characteristic function of the unit cube and

$$u_\epsilon(x, y) = v_\epsilon(y)(2\delta)^{-(n-1)/2}\mathfrak{F}_x^{-1} w\left(\frac{\xi - \bar{\xi}}{\delta}\right).$$

Then $\|u_\epsilon\| = 1$ and $\|Mu_\epsilon\| \leqq \epsilon$, contradicting (3).

PROOF OF THE THEOREM. Solving $M_\xi v = w$ is readily seen to be equivalent (via Fourier transform) to solving the Riemann-Hilbert problem

$$\Phi^-(\eta) = M(\xi, \eta)\Phi^+(\eta) + \Psi(\eta) \tag{6}$$

where $\Phi^\pm$ are sought in $(H^2_\pm(R^1))^m$, the space of transforms of L^2-vector functions supported in $R^1_\pm$, and $\Psi(\eta)$ is a given L^2-function. In the scalar case ($m=1$) this was done by Widom [5, Theorem 3.2]. It follows from Widom's results that if in (5) $\theta \neq 1/2$ and $k \geqq 0$ then M_ξ is injective and has a closed range for every $\xi \neq 0$, so that (4) is satisfied. It is clear that (4) is satisfied uniformly on the compact set $|\xi| = 1$, and by the Lemma we obtain that M is injective and has a closed range. If $\theta \neq 1/2$ and $k \leqq 0$ in (5), a consideration of M^*, the adjoint of M, which corresponds to the multiplier $\overline{M}(\xi, \eta)$, shows that M is surjective.

REMARK. It is easily seen that the expression (5) does not depend on ξ. Indeed, the homogeneity of M implies that $\lim_{\eta \to \pm\infty} M(\xi, \eta) = M(0, \pm 1)$, for any $\xi \neq 0$. This means that θ in (5) is the same for all ξ, and by continuity, the same is true for the integer k.

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