# AN EXPLICIT INVERSION FORMULA FOR FINITE-SECTION WIENER-HOPF OPERATORS[1]

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Let $T$ be the real numbers modulo 1 and $\mathfrak{A}_0$ the algebra of complex continuous functions $f(\theta)$ on $T$ which have absolutely convergent Fourier series. For $f(\theta) \in \mathfrak{A}_0$ we set

$$\|f\|_0 = \sum_{-\infty}^{\infty} | f(k) |$$

where

$$f(k) = \int_T f(\theta) e^{-2\pi i k\theta} \, d\theta.$$

For $f \in \mathfrak{A}_0$ we define

$$E^+(n) f \cdot (\theta) = \sum_{k \geqq n} f(k) e^{2\pi i k\theta},$$

$$E^-(n) f \cdot (\theta) = \sum_{k \leqq n} f(k) e^{2\pi i k\theta}.$$

DEFINITION. Let $\mathfrak{A}$ be a Banach algebra of complex continuous functions $f(\theta)$ on $T$ with norm $\|\cdot\|$. $\mathfrak{A}$ will be said to be of type $\mathfrak{M}$ if the following conditions are satisfied:

1. $\mathfrak{A}_0 \supset \mathfrak{A}$, $\|f\|_0 \leqq \|f\|$ for all $f \in \mathfrak{A}$;
2. $e^{2\pi i k\theta} \in \mathfrak{A}$ for $k = 0, \pm 1, \pm 2, \cdots$, and the trigonometric polynomials are dense in $\mathfrak{A}$;
3. there exists a constant $M$ independent of $n$ such that

$$\|E^+(n) f\| \leqq M\|f\|, \qquad \|E^-(n) f\| \leqq M\|f\|, \qquad \text{all } f \in \mathfrak{A}.$$

For $c \in \mathfrak{A}$ we define the finite-section Wiener-Hopf operators

$$W_c^+(n) f = E^+(0) E^-(n) c E^+(0) E^-(n) f,$$

$$W_c^-(n) f = E^-(0) E^+(-n) c E^-(0) E^+(-n) f.$$

Here $n \geqq 0$. Our principal result is the identities below. These identities are algebraic in character and can be seen to hold in a much more general (even in a noncommutative) setting than that considered here. We have preferred to present them in a context requiring as few definitions and as little machinery as possible.

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THEOREM 1. *Let* $\mathfrak{A} \in \mathfrak{M}$, *and let* $c(\theta) = u(\theta)^{-1}v(\theta)^{-1}$, *where* $u \in E^+(0)E^-(m)\mathfrak{A}$, $u^{-1} \in E^+(0)\mathfrak{A}$, $v \in E^-(0)E^+(-m)\mathfrak{A}$, $v^{-1} \in E^-(0)\mathfrak{A}$. *We set*

$$Y_c^+(n)f = vE^-(n)v^{-1}uE^+(0)vf,$$
$$Y_c^-(n)f = uE^+(-n)u^{-1}vE^-(0)uf.$$

*Then for* $n \geqq m$ *we have*

$$\begin{aligned} Y_c^+(n)W_c^+(n) &= W_c^+(n)Y_c^+(n) = E^+(0)E^-(n), \\ Y_c^-(n)W_c^-(n) &= W_c^-(n)Y_c^-(n) = E^-(0)E^+(-n). \end{aligned} \tag{1}$$

PROOF. We first assert that

$$\mathfrak{R}[Y_c^+(n)] \subset E^+(0)E^-(n)\mathfrak{A}, \tag{2}$$

where $\mathfrak{R}[Y_c^+(n)]$ is the range of $Y_c^+(n)$. Since $\mathfrak{R}[vE^-(n)] \subset E^-(n)\mathfrak{A}$, we have $\mathfrak{R}[Y_c^+(n)] \subset E^-(n)\mathfrak{A}$. On the other hand,

$$\begin{aligned} Y_c^+(n) &= v[I - E^+(n+1)]uv^{-1}E^+(0)v \\ &= uE^+(0)v - vE^+(n+1)uv^{-1}E^+(0)v. \end{aligned}$$

Clearly $\mathfrak{R}[uE^+(0)] \subset E^+(0)\mathfrak{A}$ and, since $n \geqq m$, $\mathfrak{R}[vE^+(n+1)] \subset E^+(1)\mathfrak{A} \subset E^+(0)\mathfrak{A}$, etc.

We have

$$\begin{aligned} Y_c^+(n)W_c^+(n) &= vE^-(n)v^{-1}uE^+(0)vE^+(0)E^-(n)u^{-1}v^{-1}E^+(0)E^-(n) \\ &= vE^-(n)v^{-1}uE^+(0)v[I - E^-(-1)]E^-(n)u^{-1}v^{-1}E^+(0)E^-(n) \\ &= vE^-(n)v^{-1}uE^+(0)vE^-(n)u^{-1}v^{-1}E^+(0)E^-(n), \end{aligned}$$

since $E^+(0) = 0$ on $\mathfrak{R}[vE^-(-1)]$. Consequently,

$$Y_c^+(n)W_c^+(n) = vE^-(n)v^{-1}uE^+(0)v[I - E^+(n+1)]u^{-1}v^{-1}E^+(0)E^-(n).$$

Consider

$$\begin{aligned} I_2 &= vE^-(n)v^{-1}uE^+(0)vE^+(n+1)u^{-1}v^{-1}E^+(0)E^-(n) \\ &= vE^-(n)v^{-1}uvE^+(n+1)u^{-1}v^{-1}E^+(0)E^-(n) \\ &= vE^-(n)uE^+(n+1)u^{-1}v^{-1}E^+(0)E^-(n) = 0. \end{aligned}$$

Here we have used $E^+(0) = I$ on $\mathfrak{R}[vE^+(n+1)]$ and $E^-(n) = 0$ on $\mathfrak{R}[uE^+(n+1)]$. Similarly,

$$\begin{aligned} I_1 &= vE^-(n)v^{-1}uE^+(0)vu^{-1}v^{-1}E^+(0)E^-(n) \\ &= vE^-(n)v^{-1}uE^+(0)u^{-1}E^+(0)E^-(n) \\ &= vE^-(n)v^{-1}uu^{-1}E^+(0)E^-(n) \\ &= vE^-(n)v^{-1}E^+(0)E^-(n), \end{aligned}$$

since $E^+(0)=I$ on $\mathcal{R}[u^{-1}E^+(0)]$. We now have

$$\begin{aligned} I_1 &= v[I - E^+(n+1)]v^{-1}E^+(0)E^-(n) \\ &= vv^{-1}E^+(0)E^-(n) - vE^+(n+1)v^{-1}E^-(n)E^+(0) = E^+(0)E^-(n), \end{aligned}$$

since $E^+(n+1)=0$ on $\mathcal{R}[v^{-1}E^-(n)]$. Combining these results we have shown that

$$Y_c^+(n)W_c^+(n) = E^+(0)E^-(n). \tag{3}$$

On the other hand[2] we find using (2) that

$$\begin{aligned} W_c^+(n)Y_c^+(n) &= E^+(0)E^-(n)u^{-1}v^{-1}E^+(0)E^-(n)vE^-(n)uv^{-1}E^+(0)v \\ &= E^+(0)E^-(n)u^{-1}v^{-1}vE^-(n)uv^{-1}E^+(0)v \\ &= E^+(0)E^-(n)u^{-1}E^-(n)uv^{-1}E^+(0)v \\ &= E^+(0)E^-(n)u^{-1}[I - E^+(n+1)]uv^{-1}E^+(0)v \\ &= E^+(0)E^-(n)u^{-1}uv^{-1}E^+(0)v \\ &= E^+(0)E^-(n)v^{-1}E^+(0)v, \end{aligned}$$

since $E^-(n)=0$ on $\mathcal{R}[u^{-1}E^+(n+1)]$. Thus

$$\begin{aligned} W_c^+(n)Y_c^+(n) &= E^-(n)E^+(0)v^{-1}[I - E^-(-1)]v \\ &= E^-(n)E^+(0)v^{-1}v = E^+(0)E^-(n). \end{aligned} \tag{4}$$

Here we have used $E^+(0)=0$ on $\mathcal{R}[v^{-1}E^-(-1)]$.

The second relation in (1) can be proved in exactly the same way.

These identities have many applications. They can be used to give a new proof of the norm inequality for the finite-section Wiener-Hopf operator of [3], and a new and much simpler proof of the "#" inequality for the finite-section Wiener-Hopf operators defined on compact groups with ordered duals of [5] and [6]. We will content ourselves with one application. We assume the reader is familiar with the theory developed in [1]-[3] and [4].

Let $c(\theta)\in\mathcal{A}$. We define the infinite Wiener-Hopf operators $W_c^+$ and $W_c^-$ by

$$\begin{aligned} W_c^+f &= E^+(0)cE^+(0)f, \\ W_c^-f &= E^-(0)cE^-(0)f. \end{aligned}$$

We say that $c\in \mathrm{WH}(\mathcal{A})$ if $W_c^+$ restricted to $E^+(0)\mathcal{A}$ and $W_c^-$ restricted to $E^-(0)\mathcal{A}$ have (bounded) inverses. A necessary and sufficient condition that $c(\theta)\in\mathrm{WH}(\mathcal{A})$ is that

[2] Since $E^+(0)E^-(n)\mathcal{A}$ is finite dimensional, (2) and (3) together imply (4). However, in more general situations (4) must be verified independently.

$$c(\theta) = d^2 U(\theta)^{-1} V(\theta)^{-1},$$

where $U,\ U^{-1} \in E^+(0)\mathfrak{A},\ V,\ V^{-1} \in E^-(0)\mathfrak{A}$, and

$$\int_T U(\theta)\, d\theta = \int_T V(\theta)\, d\theta = d^2.$$

THEOREM 2. *If $c(\theta) \in \mathrm{WH}(\mathfrak{A})$, then there exists an integer $N$ such that, if $n \geqq N$,*

$$\begin{aligned} W_c^+(n) X_c^+(n) &= X_c^+(n) W_c^+(n) = E^+(0) E^-(n), \\ W_c^-(n) X_c^-(n) &= X_c^-(n) W_c^-(n) = E^-(0) E^+(-n), \end{aligned} \tag{5}$$

*where*

$$\begin{aligned} X_c^+(n) &= d(n)^{-2} v(n) E^-(n) v(n)^{-1} u(n) E^+(0) v(n), \\ X_c^-(n) &= d(n)^{-2} u(n) E^+(-n) u(n)^{-1} v(n) E^-(0) u(n). \end{aligned}$$

*Here $u(n, \theta)$ and $v(n, \theta)$ are defined by*

$$\begin{aligned} u(n, \theta) &\in E^+(0) E^-(n) \mathfrak{A}, \qquad & W_c^+(n) u(n) &= 1, \\ v(n, \theta) &\in E^-(0) E^+(-n) \mathfrak{A}, \qquad & W_c^-(n) v(n) &= 1, \end{aligned}$$

*and*

$$d(n)^2 = \int_T u(n, \theta)\, d\theta = \int_T v(n, \theta)\, d\theta.$$

The simple proof of this result is omitted.

We have not offered any indication of the origin of (1). It is possible to derive (1) from known results but to do so obscures its usefulness which lies in the fact it can be proved directly. However, we will point out that it is related to the identities of [**4**, §5] and [**6**, §8].

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