# TAMING CANTOR SETS IN $E^n$

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1. **Introduction.** Any two compact, perfect, zero-dimensional and nondegenerate metric spaces are homeomorphic. We call such a space a *Cantor set.* A Cantor set $C$ in Euclidean space $E^n$ is called *tame* if there is a homeomorphism $h$ of $E^n$ onto $E^n$ such that $h(C) \subset E^1 \times \{0_{n-1}\} = E^1 \subset E^n$. For examples of *wild* (i.e., nontame) Cantor sets, see [1], [4], [3], and [9]. The examples of Blankenship [3] give the existence of wild Cantor sets in $E^n$ for each $n \geqq 3$.

Homma [8] and Bing [2, Theorem 5.1] have shown that a Cantor set $C$ in $E^3$ is tame if and only if $E^3 - C$ is 1-ULC (definition below). It is our purpose here to extend this useful characterization to Cantor sets in $E^n$ $(n \neq 4)$. We assume the customary metric on $E^n$ throughout this paper. Let $K$ be a compact set in $E^n$. Then we say that $E^n - K$ is 1-ULC if for each $\epsilon > 0$ there is a $\delta > 0$ such that each loop (i.e., closed curve) of diameter less than $\delta$ in $E^n - K$ is null-homotopic in $E^n - K$ on a set of diameter less than $\epsilon$.

We sketch the proof below, relying heavily on the *cellularity criterion* [10, Theorems 1 and 1′]. For $n \geqq 5$, this criterion implies that a compact absolute retract $X$ in the interior of a piecewise-linear (abbreviated pwl) $n$-manifold $M^n$ is cellular with respect to piecewise-linear cells if and only if for each open set $U \subset M$ containing $X$ there is an open set $V$ such that $X \subset V \subset U$ and each loop in $V - X$ is null-homotopic in $U - X$.

2. **The theorem.** We first state some lemmas. For Lemma 1, see [11, Theorem 3], [12, Theorem 4], and [6, Theorem 3]. In Whitehead's theorem [12], we take $K = \text{Bd } M$.

LEMMA 1. *Let $M^n$ be a compact piecewise-linear n-manifold (possibly with boundary), and let $E_1$ and $E_2$ be piecewise-linear n-cells in* Int $M$. *Then there is a piecewise-linear homeomorphism $h: M \to M$ such that $h(E_1) = E_2$ and $h|$*Bd $M$ *= the identity.*

LEMMA 2. *Let $C$ be a Cantor set in $E^n$, $n \geqq 3$. Then $C$ is tame if for each $\epsilon > 0$ there is a finite, disjoint collection of piecewise-linear n-cells, each of diameter less than $\epsilon$, whose interiors cover $C$.*

The proof of Lemma 2 is essentially the same as in the three-di-

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mensional case (see [8] and [2, Theorem 1.1]). One uses Lemma 1 to obtain the extensions of homeomorphisms required in the proof.

It follows from [10, Theorem 6] that if $n\neq 4$ and $A$ is a cellular arc in $E^n$, then each subarc of $A$ is cellular with respect to piecewise-linear cells. Since, for given $\epsilon>0$, each Cantor set in such an $A$ can be covered by a finite, disjoint collection of arcs each lying in $A$ and each having diameter less than $\epsilon$, Lemma 2 gives the following.

LEMMA 3. *Let $A$ be a cellular arc in $E^n$, $n\neq 4$. Then each Cantor set in $A$ is tame in $E^n$.*

THEOREM. *Let $C$ be a Cantor set in $E^n$, $n\neq 4$. Then $C$ is tame if and only if $E^n-C$ is* 1-ULC.

PROOF. The "only if" half is clear. Suppose now that $E^n-C$ is 1-ULC. By previous remarks, we may assume that $n\geqq 5$. It is not difficult to show that there is an arc $A$ such that $C\subset A\subset E^n$ and $A$ is locally polyhedral except possibly at points of $C$. By Lemma 3, we have only to verify that the hypotheses of the cellularity criterion are satisfied for $A$. In fact, we prove the stronger assertion (see [10, Theorem 5]) that $E^n-A$ is 1-ULC.

Toward this end, let $\epsilon>0$ be given. Choose $\delta>0$ so that each loop in $E^n$ of diameter less than $\delta$ is null-homotopic in $E^n$ on a set of diameter less than $\epsilon$. Now if a pwl loop in $E^n-A$ of diameter less than $\delta$ is given, it may be contracted in a pwl manner in $E^n$ on a set of diameter less than $\epsilon$. Since $E^n-C$ is 1-ULC, the contraction may be altered slightly so as to be pwl and to take place in $E^n-C$ (see [5, Theorem 2] and [7, Lemma 2]). Since $A$ is locally a one-dimensional polyhedron away from $C$ and $n\geqq 4$, the contraction may be altered slightly again so as to completely miss $A$. This completes the proof.

REMARK. If $n=4$, the proof above still shows that $C$ lies on an arc whose complement is 1-ULC.

COROLLARY 1. *Let $\Sigma^k$ be a $k$-sphere, $k\leqq n-1$, topologically embedded in $E^n$ in a locally nice manner, in the sense of [7], where $n\geqq 5$. Then each Cantor set in $\Sigma$ is tame in $E^n$.*

PROOF. For, [7, Theorem 1] states that each compact absolute retract in $\Sigma$ is cellular in $E^n$. Lemma 3 thus may be applied.

COROLLARY 2. *Let $C$ be a Cantor set in $E^m$ and $Z$ a compact* 0*-dimensional set in $E^n$, where $m, n\geqq 1$. Then the Cantor set $C\times Z$ is tame in $E^m\times E^n=E^{m+n}$.*

PROOF. If $m=n=1$ or $m=n=2$, the result follows from the fact that each Cantor set in $E^2$ is tame. If at least one of $m$ and $n$ is at least two and $m+n\neq 4$, we can appeal to the proof of [**10**, Theorem 7] for the fact that $E^{m+n}-C\times Z$ is 1-ULC and hence, by the present Theorem, $C\times Z$ is tame in $E^{m+n}$. If (say) $m=3$ and $n=1$, we use Lemma 2 directly. See the first paragraph of the proof of [**10**, Theorem 8] for references.

REMARK. The above Theorem and its corollaries can, of course, be stated and proved with *any* compact 0-dimensional subset of $E^n$ replacing the Cantor set.

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