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A SPARSE REGULAR SEQUENCE OF EXPONENTIALS CLOSED ON LARGE SETS

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Introduction. For a given sequence $\{\lambda_k\}$ of complex numbers, the problem of determining those intervals I on which the exponentials $\{e^{i\lambda_k x}\}$ are complete in various function spaces has been extensively studied [3]. Since the problem is invariant under a translation of I, only the lengths of I are involved, and attention has focused on the relation between these lengths and the density of the sequence $\{\lambda_k\}$. With the function space taken to be $L^p(I)$ for $1 \leqq p < \infty$, or $C(I)$, the continuous functions on I, the general character of the results has been that there exist sparse real sequences ($\lim r^{-1}$ (the number of $|\lambda_k| < r) = 0$, for example) for which I can be arbitrarily long [2], but all such sequences are nonuniformly distributed; when a sequence is sufficiently regular, in the sense that λ_k is close enough to k, the length of I cannot exceed 2π [4, p. 210]. Most recently, in a complete solution which accounts for all these phenomena, Beurling and Malliavin have proved that the supremum of the lengths of I is proportional to an appropriately defined density of $\{\lambda_k\}$ [1].

The purpose of this note is to show that the situation is quite different when the single interval I is replaced by a union of intervals. Specifically, we will construct a real symmetric (or positive) sequence $\{\lambda_k\}$ arbitrarily close to the integers, for which the corresponding exponentials are complete in $C(S)$, where S is any finite union of the intervals $|x-2n\pi| < \pi-\delta$, with integer n and $\delta > 0$, and so has arbitrarily large measure. Thus, for sets S more general than intervals, it would seem that no relation can be expected between measure of S and density of $\{\lambda_k\}$.

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Results.

LEMMA 1. *We may partition the positive integers into an infinite number of disjoint sequences* $S_r=\{k_n^{(r)}\}_{n=1}^{\infty}$, $r=1, 2, \cdots$, *with the property that* $\limsup n/k_n^{(r)}=1$ *for each* r.

PROOF. We will define S_r as the union $\bigcup_{i=1}^{\infty}\sigma_{i,r}$ of disjoint blocks $\sigma_{i,r}$ of consecutive integers. To define $\sigma_{i,r}$, we order the integer couples (i, r) with $i, r\geqq 1$, by increasing values of $s=i+r$, and for same values of s, by increasing i. We let $\sigma_{1,1}=\{1\}$ and choose the remaining $\sigma_{i,r}$ consecutively in the order of the (i, r), letting each $\sigma_{p,q}$ begin with the first integer not included in the previously defined σ; we pick $\sigma_{p,q}$ so long that if N is the number of integers in $\sigma_{p,q}$, k is the first of them, and M is the total number of integers in the (already determined) $\sigma_{j,q}$ with $j<p$, then $(N+M)/(k+N-1)>1-1/p$. By this construction, whenever $k_n^{(r)}$ in S_r coincides with the right-hand endpoint of a $\sigma_{i,r}$ we have $n/k_n^{(r)}>1-1/i$, so that $\limsup n/k_n^{(r)}=1$ for each r. Finally, the S_r are all disjoint and their union is all positive integers. Lemma 1 is established.

LEMMA 2. *With* $\theta_1, \theta_2, \cdots$ *real numbers, set* $z_k=e^{i2\pi\theta_k}$, *and denote by* $\Delta(\theta_1, \cdots, \theta_n)$ *the determinant whose* $2j$*th row is* $z_j^{n-1}, z_j^{n-2}, \cdots, z_j^{-n}$ *and whose* $(2j-1)$*th row is* $z_j^{-n+1}, z_j^{-n+2}, \cdots, z_j^{n}$, *with* $1\leqq j\leqq n$. *Then given* $\epsilon>0$ *we may choose* $\theta_1, \theta_2, \cdots$ *with* $|\theta_i|<\epsilon$ *so that, for all* n, *we have* $\Delta(\theta_1, \cdots, \theta_n)\neq 0$.

PROOF. The condition $|\theta_i|<\epsilon$ is equivalent to $z_i\in\gamma$, with γ an appropriate arc of $|z|=1$. First, letting z_1 be any point of γ other than $z=1$ ensures $\Delta(\theta_1)\neq 0$. Then we observe that $\Delta(\theta_1, \cdots, \theta_n)$ can be expanded as a polynomial in z_n and z_n^{-1}, with leading coefficient $\Delta(\theta_1, \cdots, \theta_{n-1})$. Assuming $z_1, \cdots, z_{n-1}$ have been chosen to satisfy the requirements of the lemma, this coefficient does not vanish, and so $\Delta(\theta_1, \cdots, \theta_n)$ considered as a function of z_n is not identically zero; being analytic in z_n it therefore cannot vanish everywhere for z_n on γ. Thus we may find a point $e^{i2\pi\theta_n}\in\gamma$ such that when $z_n=e^{i2\pi\theta_n}$, $\Delta(\theta_1, \cdots, \theta_n)\neq 0$. By induction, Lemma 2 is established.

THEOREM. *Given* $\epsilon>0$, *there exists a symmetric real sequence* $\{\lambda_k\}_{-\infty}^{\infty}$ *with* $|\lambda_k-k|<\epsilon$ *such that the functions* $\{e^{i\lambda_k x}\}$ *are complete in continu-*

ous functions on every finite union of the intervals $|x-2n\pi|<\pi-\delta$, *with integer* n *and* $\delta>0$.

PROOF. We will partition the integers into disjoint subsets, shift each subset by a small amount, and let the sequence $\{\lambda_k\}$ consist of the points so obtained. Then we will show that completeness of the corresponding exponentials on unions of certain intervals is equivalent to completeness on a single interval of $\{e^{ikx}\}$, with k in one subset, and thereby reduce the theorem to a classical result.

Let S_r, $r=1, 2, \cdots$, be the disjoint subsets of the integers defined in Lemma 1, and let $S_{-r}=\{k \mid -k\in S_r\}$. Similarly, let θ_r, $r=1, 2, \cdots$, be the numbers constructed in Lemma 2, and let $\theta_{-r}=-\theta_r$. Now for $k\in S_r$, $r=\pm 1, \pm 2, \cdots$, set $\lambda_k=k+\theta_r$, and $\lambda_0=0$. Then the sequence $\{\lambda_k\}_{-\infty}^{\infty}$ is symmetric and $|\lambda_k-k|<\epsilon$.

To prove the theorem we must show that given N and $\delta>0$, the exponentials $\{e^{i\lambda_k x}\}$ are complete in $C(S)$, where $S=\bigcup_{n=-N+1}^{N} I_n$, and I_n is the interval $|x-2n\pi|<\pi-\delta$, or equivalently [4, p. 115] that any bounded measure supported on S which annihilates these exponentials must vanish identically. Accordingly, let $\mu(x)$ be such a measure, and denote by $\mu_n(x-2n\pi)$ the restriction of $\mu(x)$ to I_n. Then $\mu_n(x)$ is a bounded measure supported on I_0, and

$$\mu(x) = \sum_{n=-N+1}^{N} \mu_n(x-2n\pi). \tag{1}$$

Now by a change of variable,

$$\int_S e^{i\lambda_k x} d\mu(x) = \sum_{n=-N+1}^{N} e^{i\lambda_k 2n\pi} \int_{I_0} e^{i\lambda_k x} d\mu_n(x),$$

and if $k\in S_r$, $e^{i\lambda_k 2n\pi}=e^{i\theta_r 2n\pi}$ and does not depend on k. Thus if $\mu(x)$ annihilates the exponentials $\{e^{i\lambda_k x}\}$ for $k\in S_r$, so does

$$\sum_{n=-N+1}^{N} e^{i\theta_r 2n\pi}\mu_n(x), \tag{2}$$

which is a bounded measure supported on the single interval I_0.

We now invoke a known result [3, p. 13]: since by Lemma 1, $\limsup n/k_n=1$ in each S_r, the set S_r has Polya density 1, and so the exponentials $\{e^{ikx}\}$ for $k\in S_r$ are complete in continuous functions on any interval of length less than 2π, in particular on I_0. By definition of the set $\{\lambda_k\}$ for $k\in S_r$ as a translate of S_r, the same is true of the exponentials $\{e^{i\lambda_k x}\}$, $k\in S_r$, and consequently the measure (2) on I_0 which annihilates them must vanish identically. We conclude

$$\sum_{n=-N+1}^{N} e^{i\theta_r 2n\pi}\mu_n(x) \equiv 0, \qquad x \in I_0, \tag{3}$$

for each r. Writing (3) with $r = \pm 1, \cdots, \pm N$ yields a system of $2N$ linear equations for the $2N$ measures $\mu_n(x)$, $-N+1 \leqq n \leqq N$, whose determinant is precisely $\Delta(\theta_1, \cdots, \theta_N)$ and so, by Lemma 2, does not vanish. Thus the only solution to this system is $\mu_n(x) \equiv 0$, $-N+1 \leqq n \leqq N$, whence $\mu(x) \equiv 0$ by (1). This completes the proof of the theorem.

Remarks. 1. By an obvious modification of the proof, the exponentials $\{e^{i\lambda_k x}\}$ with $k \in S_r$ for $r > 0$ have the same completeness property.

2. We may give a constructive proof of the theorem along the same lines. Shifting each I_n to I_0 transforms the problems of approximating a continuous function on S by linear combinations of the exponentials $\{e^{i\lambda_k x}\}$ into that of solving a system of linear equations on I_0 with nonzero determinant, and thereby again reduces matters to approximating on I_0 by linear combinations of $\{e^{ikx}\}$ for $k \in S_r$.

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