THE STABLE STRUCTURE OF QUITE GENERAL LINEAR GROUPS

BY HYMAN BASS

Communicated by R. Bott, December 28, 1963

1. **Introduction.** Dieudonné [5] determined the normal subgroups of $\mathrm{GL}(n, A)$ for an (even noncommutative) field, A, and Klingenberg recently showed [7; 8] that Dieudonné's result, suitably formulated, survives without surprises for A any local ring. The results described here constitute the beginnings of a global theory. The information they yield on $\mathrm{SL}(n, \boldsymbol{Z})$, combined with a rather formidable cohomological calculation, is the basis of the proof in [3] that every subgroup of finite index in $\mathrm{SL}(n, \boldsymbol{Z})$, $n \geqq 3$, contains a congruence subgroup.

This material, the details of which will appear in [1], is based on the algebraic K-theory described in [4]. The topological intuition thereby afforded intervenes via the space, X, of maximal ideals of a commutative ring, A. Thus, our results on $\mathrm{GL}(n, A)$ are effective only if n is sufficiently large compared with dim X, i.e. only if n is in the stable range. If A is semi-local this is no restriction, X being then finite. If $A = \boldsymbol{Z}$ then dim $X = 1$. For general A we must let n go to infinity (stabilize) (§2). While the Dieudonné-Klingenberg theorem may fail even then, its failure is measured by certain abelian groups, $K^1(A, \mathfrak{q})$, one for each ideal $\mathfrak{q}$. When $\mathfrak{q} = A$ we write $K^1(A) = K^1(A, A)$. When dim $X = 0$ they reduce to something essentially trivial, and we recover Dieudonné-Klingenberg.

In a joint paper with A. Heller and R. Swan [2] the homomorphisms $K^1(A) \to K^1(A[t])$ and $K^1(A) \to K^1(A[t, t^{-1}])$, t an indeterminate, are analyzed (see §5). Concerning the latter Atiyah has pointed out that our result is an analogue of Bott periodicity for the unitary group (see §6).

Finally, various of these results yield information (§7) on J. H. C. Whitehead's groups of simple homotopy types [9], results which extend some earlier work of G. Higman [6].

2. **Stable structure theorem.** For a ring A, $\mathrm{E}(n, A)$ denotes the subgroup of $\mathrm{GL}(n, A)$ generated by all elementary matrices, i.e. those differing from the identity in a single, off diagonal, coordinate. If $\mathfrak{q}$ is an ideal we write

$$\mathrm{GL}(n, A, \mathfrak{q}) = \ker(\mathrm{GL}(n, A) \to \mathrm{GL}(n, A/\mathfrak{q})),$$

and $\mathrm{E}(n, A, \mathfrak{q})$ denotes the *normal* subgroup of $\mathrm{E}(n, A)$ generated by the elementary matrices in $\mathrm{GL}(n, A, \mathfrak{q})$.

We further identify $\alpha \in \mathrm{GL}(n, A)$ with

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1, A),$$

and thus write $\mathrm{GL}(A, \mathfrak{q}) = \bigcup_n \mathrm{GL}(n, A, \mathfrak{q})$ and $\mathrm{E}(A, \mathfrak{q}) = \bigcup_n \mathrm{E}(n, A, \mathfrak{q})$. When $\mathfrak{q} = A$ we abbreviate, $\mathrm{GL}(A) = \mathrm{GL}(A, A)$ and $\mathrm{E}(A) = \mathrm{E}(A, A)$.

THEOREM. *Let A be any ring.*

(a) *For all ideals*, $\mathfrak{q}$,

$$\mathrm{E}(A, \mathfrak{q}) = [\mathrm{E}(A), \mathrm{E}(A, \mathfrak{q})] = [\mathrm{GL}(A), \mathrm{GL}(A, \mathfrak{q})].$$

(b) *If $H \subset \mathrm{GL}(A)$ is normalized by* $\mathrm{E}(A)$ *then for a unique ideal*, $\mathfrak{q}$, $\mathrm{E}(A, \mathfrak{q}) \subset H \subset \mathrm{GL}(A, \mathfrak{q})$, *and H is then* (*by* (a)) *normal in* $\mathrm{GL}(A)$.

This theorem tells us that a knowledge of the normal subgroups of $\mathrm{GL}(A)$ is equivalent to a determination of the abelian groups

$$K^1(A, \mathfrak{q}) = \mathrm{GL}(A, \mathfrak{q})/\mathrm{E}(A, \mathfrak{q}).$$

When $\mathfrak{q} = A$

$$K^1(A) = \mathrm{GL}(A)/\mathrm{E}(A)$$

is just the commutator quotient of $\mathrm{GL}(A)$.

3. The stable range.

THEOREM. *Suppose A is an algebra, finitely generated as a module, over a commutative ring whose maximal ideal spectrum is a Noetherian space of dimension d.*

For $n > d+1$ and for all ideals, $\mathfrak{q}$:

(a) $\mathrm{E}(n, A, \mathfrak{q})$ *is normal in* $\mathrm{GL}(n, A)$, *and* $\mathrm{GL}(n, A, \mathfrak{q}) = \mathrm{GL}(d+1, A, \mathfrak{q})\, \mathrm{E}(n, A, \mathfrak{q})$.

Suppose $n > \max(d+1, 2)$:

(b) $\mathrm{E}(n, A, \mathfrak{q}) = [\mathrm{E}(n, A), \mathrm{E}(n, A, \mathfrak{q})] = [\mathrm{E}(n, A), \mathrm{GL}(n, A, \mathfrak{q})]$ *for all ideals*, $\mathfrak{q}$.

(c) *If $H \subset \mathrm{GL}(n, A)$ is normalized by* $\mathrm{E}(n, A)$ *then there is a unique ideal*, $\mathfrak{q}$, *such that* $\mathrm{E}(n, A, \mathfrak{q}) \subset H$ *and the image of H in* $\mathrm{GL}(n, A/\mathfrak{q})$ *lies in the center.*

For $n \geqq \max(2(d+1), 3)$, *and for all ideals*, $\mathfrak{q}$:

(d) $\mathrm{E}(n, A, \mathfrak{q}) = [\mathrm{GL}(n, A), \mathrm{GL}(n, A, \mathfrak{q})]$.

There are some technical inadequacies in this theorem, and one fundamental deficiency, which is best described by considering the homomorphisms,

$$\mathrm{GL}(n, A, \mathfrak{q})/\mathrm{E}(n, A, \mathfrak{q}) \xrightarrow{f_n} \mathrm{GL}(n+1, A, \mathfrak{q})/\mathrm{E}(n+1, A, \mathfrak{q}).$$

These define a direct system of eventually abelian groups whose limit is $K^1(A, \mathfrak{q})$. Part (a) says f_n is surjective for $n \geqq d+1$, and topological considerations suggest the

CONJECTURE. *f_n is injective for $n > d+2$.* The affirmation of this conjecture, which would have a number of important applications, constitutes, when A is a division algebra, the essential part of Dieudonné's theory of noncommutative determinants [5].

4. Finiteness theorems.

THEOREM. *Let Σ be a semi-simple, finite-dimensional algebra over $\boldsymbol{Q}$ with q simple factors, and suppose $\boldsymbol{R} \otimes_{\boldsymbol{Q}} \Sigma$ has r simple factors. If A is an order in Σ and $\mathfrak{q}$ an ideal in A, then* $\ker(K^1(A, \mathfrak{q}) \to K^1(\Sigma))$ *is finite, and $K^1(A, \mathfrak{q})$ is a finitely generated abelian group of rank $\leqq r-q$, with equality when $A/\mathfrak{q}$ is finite.*

THEOREM. *Suppose Σ above is simple, and let* $\mathrm{SL}(n, A)$ *denote the elements of reduced norm one in* $\mathrm{GL}(n, A)$. *Then center* $\mathrm{SL}(n, A) =$ center $\mathrm{E}(n, A)$ *is finite. Moreover:*

(a) *For $n \geqq 3$ a normal subgroup of* $\mathrm{E}(n, A)$ *is either finite (and central), or of finite index.*

(b) *For all sufficiently large n the same is true of* $\mathrm{SL}(n, A)$.

REMARKS. 1. The theorem of §3 applies to A here with $d=1$. Hence, the conjecture there alleges that $n \geqq 3$ suffices in (b) above.

2. For $A = \boldsymbol{Z}$, $\mathrm{SL}(n, \boldsymbol{Z}) = \mathrm{E}(n, \boldsymbol{Z})$, since $\boldsymbol{Z}$ is euclidean.

5. Polynomial and related extensions.

The results announced here are from a joint paper with A. Heller and R. Swan, [2].

For any ring A we denote the Grothendieck group of finitely generated projective right A-modules by $K^0(A)$.

THEOREM. *Let A be a ring and t an indeterminate. There are natural split exact sequences,*

$$0 \to U_0 \to K^1(A[t]) \to K^1(A) \to 0$$

and

$$0 \to U \to K^1(A[t, t^{-1}]) \to K^0(A) \oplus K^1(A) \to 0.$$

Here U denotes the image of matrices of the form $1 + (t^{\pm 1} - 1)\alpha$, with α nilpotent over A.

We call A *right regular* if A is right Noetherian and if every finitely generated right A-module has a finite projective resolution. By the Syzygy Theorem regularity of A is inherited by $A[t]$, and hence also by its ring of quotients, $A[t, t^{-1}]$.

THEOREM. *If A is right regular then all unipotents in* GL(A) *are trivial in* $K^1(A)$.

COROLLARY. *If A is right regular and $t_1, \cdots, t_n$ are indeterminates then $K^i(A[t_1, \cdots, t_n]) = K^i(A)$, $i=0, 1$. (For $i=0$ this result is due to Grothendieck.)*

COROLLARY. *Let π be a free abelian group of rank n, and let A be a right regular ring. Then*

$$K^0(A\pi) = K^0(A), \quad \textit{and} \quad K^1(A\pi) = K^0(A)^n + K^1(A).$$

6. **Relation to Bott's theorem.** Let X be a finite CW-complex, and let $A = \mathbf{C}(X)$, the ring of complex continuous functions on X. If $t: X \times S^1 \to \mathbf{C}$ sends (x, z) to z we have $A[t, t^{-1}] \subset \mathbf{C}(X \times S^1)$. Let $(A^*)^0$ denote the group of functions $X \to \mathbf{C}^*$ homotopic to a constant.

THEOREM. *There is a commutative diagram,*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U & \to & U \oplus (A^*)^0 & \to & (A^*)^0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U \oplus K^0(A) & \to & K^1(A[t, t^{-1}]) & \to & K^1(A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K^0(X) & \to & K^{-1}(X \times S^1) & \to & K^{-1}(X) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns.

The middle row here comes from the theorem of §5. Exactness of the bottom row is the unitary periodicity theorem. The ideas of the recent proof of Atiyah-Bott can be used to derive the latter from the former.

7. **Groups of simple homotopy types.** If π is a group, we write $\mathrm{Wh}(\pi) = K^1(\mathbf{Z}\pi)/\pm\pi$, meaning $K^1(\mathbf{Z}\pi)$ reduced modulo the image of $\pm\pi \subset \mathrm{GL}(1, \mathbf{Z}\pi) \subset \mathrm{GL}(\mathbf{Z}\pi)$. J. H. C. Whitehead's simple homotopy types [9] are topological invariants which live in these groups. From §4 we have:

THEOREM. *Let π be a finite group with q irreducible rational representations and r irreducible real representations. Then* Wh(π) *is a finitely generated abelian group of rank $r-q$.*

It is known (Artin, Witt) that q is the number of conjugacy classes of cyclic subgroups, and r is the number of $\sim$ classes, where $a \sim b$ in π means a is conjugate to $b^{\pm 1}$.

EXAMPLES. 1. For π finite abelian $r = q \Leftrightarrow \pi$ has exponent 4 or 6. C. Higman has shown [6] that $\mathrm{Wh}(\pi) = 0$ if $[\pi: 1] \leqq 4$. Using the result of [3] one can show $\mathrm{Wh}(\pi) = 0$ for π of type $(2, 2, \cdots)$.

2. If $\boldsymbol{Q}$ is a splitting field for π then $r = q$. This case includes the symmetric groups. $r = q$ also for the quaternion group.

3. If $\pi = \boldsymbol{Z}/n\boldsymbol{Z}$ then q is the number of divisors of n, and $r = [n/2] + 1$.

From §6 we have:

THEOREM. *If π is free abelian* $\mathrm{Wh}(\pi) = 0$.

REFERENCES

1. H. Bass, *K-theory and stable algebra*, Inst. Hautes Études Sci. Publ. Math.

2. H. Bass, A. Heller and R. Swan, *The Whitehead group of a polynomial extension*, (to appear).

3. H. Bass, M. Lazard and J.-P. Serre, *Sous groupes d'indice fini dans* $\boldsymbol{SL}(n, \boldsymbol{Z})$, Bull. Amer. Math. Soc. **70** (1964), 385–392.

4. H. Bass and S. Schanuel, *The homotopy theory of projective modules*, Bull. Amer. Math. Soc. **68** (1962), 425–428.

5. J. Dieudonné, *Les déterminants sur un corps non commutatif*, Bull. Soc. Math. France **71** (1943), 27–45.

6. G. Higman, *The units of group rings*, Proc. London Math. Soc. **46** (1940), 231–248.

7. W. Klingenberg, *Lineare Gruppen über lokalen Ringen*, Amer. J. Math. **83** (1961), 137–153.

8. ———, *Die Struktur der linearen Gruppe über einem nichtkommutatives lokalen Ring*, Arch. Math. **13** (1962), 73–81.

9. J. H. C. Whitehead, *Simple homotopy types*, Amer. J. Math. **72** (1950), 1–57

COLUMBIA UNIVERSITY