INFINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF COMPACT SEMI-SIMPLE GROUPS

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A great deal of progress has been made in the past two decades in the study, and especially the classification of, the unitary representations of the open semi-simple Lie groups. On the other hand, the classification problem is not so well posed or effectively definitive as might initially appear. The existing classifications consist roughly and for the most part of lists of inequivalent irreducible representations exhausting those in a given abstractly defined or otherwise cohesive category. Such lists can of course be extremely useful. However, the form in which the representations are explicitly given, the choice of concrete representation space, etc., is neither effectively unique nor immaterial.

For example, the work of Kunze and Stein giving a highly compact description of the representations in certain categories by means of analytic continuation in relevant parameters is based on a different presentation from that in the earlier literature on the representations in question. A different aspect which may be cited is that most classifications may be regarded as based on the existence of a maximal abelian algebra of operators left invariant by the representation. From this derives a representation in terms of the action of the group as a transformation group on the spectrum of a dense subalgebra, combined with a corresponding "multiplier." However, the question of the extent to which there exist other such algebras of imprimitivity, apart from the ones involved in the existing presentations, is largely unanswered.

On the whole, the general structure of the representations of the open semi-simple Lie groups has not yet been shown to possess the transparency and unique form which might be hoped for. The purpose of the present note is to describe an observation indicating possibilities for the classification of these representations directly in terms of local representations of the associated compact groups. It is of course too much to expect that the unitary representations of the compact groups should suffice; grossly speaking, there are simply too few of them. On the other hand, there is an apparently prevalent conception that there are no other interesting ones in Hilbert space, based on results indicating the similarity of various types of analyti-

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cally well-behaved representations of compact groups to unitary ones. The fact is however that a compact semi-simple group can have infinite-dimensional irreducible local representations by operators in Hilbert space, and that certain of these determine in a direct way the irreducible unitary representations of the associated complex group. Similar results are applicable to the other real forms. These results lend substance to the hope that the set of all analytically tractable irreducible local representations in Hilbert space of a given compact semi-simple group may form in a natural way a connected manifold,[2] from which the irreducible representations of the other forms of the group may be determined by a variant of Weyl's "unitarian trick-" a somewhat paradoxical variant, since it depends on the use of representations of the compact form which are not completely reducible.

These representations are not analytically pathological. To facilitate explicitness on this aspect, define a *domain of control* for a closed operator T as a domain $\mathfrak{D}$ which is contained in its domain and is such that the closure of $T|\mathfrak{D}$ is T. Next define an *analytic representation* of a Lie group G in a Hilbert space $\mathfrak{H}$ as a local map V from a group nucleus into closed densely-defined operators in $\mathfrak{H}$ admitting a common dense domain of control $\mathfrak{D}$, with the following properties: (i) there is a representation v_0 of the Lie algebra $\mathfrak{g}$ of G by operators in $\mathfrak{H}$ with domain $\mathfrak{D}$, whose representation operators leave $\mathfrak{D}$ invariant and have adjoints defined on $\mathfrak{D}$ likewise leaving it invariant, from which V may be recovered by the equation

$$V(e^X)z = \sum_n \frac{(v_0(X))^n}{n!} z, \qquad z \in \mathfrak{D},$$

where it is assumed that the series converges absolutely, as a power series in the coordinates of X, with coefficients in $\mathfrak{H}$, near $X=0$; (ii) the same is true of the contragredient representation $e^X \to V(e^{-X})^*$ in relation to $v_0^*|\mathfrak{D}$. Such a domain $\mathfrak{D}$ may be called an *analytic domain* for the representation V; v_0 may be called an *associated infinitesimal representation* for V; and the representation v defined by the equation $v(X) = v_0(X)^{**}$ may be called the *maximal infinitesimal representation* for V (since the domains of the $v(X)$ vary with X, this is a representation in the sense of the usual calculus of partially defined operators rather than in the elementary algebraic sense). It may be noted that V and its contragredient representation are necessarily local representations in the usual sense that $V(ab)z = V(a)V(b)z$ for z

[2] Quite possibly an algebraic one connected with the center of the enveloping algebra.

in $\mathfrak{D}$ and a and b sufficiently close to the unit, and the same with $V(a)$ replaced by $V(a^{-1})^*$, by the absolute convergence of the power series involved (cf. [2, p. 601]).

A representation is called *holomorphic* or *anti-holomorphic* if it is analytic and if its associated infinitesimal representations are complex linear or anti-linear, in the usual algebraic sense (of course, this definition is relative to a given complex structure in the Lie algebra). A closed linear subspace $\mathfrak{M}$ of $\mathfrak{H}$ is called **-invariant* under a set of closed densely-defined operators in $\mathfrak{H}$ in case the projection with range $\mathfrak{M}$ commutes with them in the sense of von Neumann. (If a set of bounded operators is closed under adjunction, invariance under them in this sense is equivalent to ordinary invariance, but otherwise it is in principle stronger, in that the orthocomplementary manifold is required to be invariant also.)

The basic result may be given as

THEOREM 1. *Let V be an analytic representation of the complex Lie group G on the complex Hilbert space $\mathfrak{H}$ with analytic domain $\mathfrak{D}$. Then there exist unique holomorphic and anti-holomorphic representations R and S of G with analytic domain $\mathfrak{D}$ which commute and have product V on $\mathfrak{D}$:*

$$R(a)S(b)z = S(b)R(a)z, \quad R(a)S(a)z = V(a)z \quad (z \in \mathfrak{D};\ a,\ b \text{ near } e).$$

Conversely, any given pair of representations R and S having the cited properties (except the last) have as their product an analytic representation V from which they derive in the indicated fashion.

For the proof, observe that on the algebraic side, if v is any representation of a complex Lie algebra $\mathfrak{G}$, then the equations

$$r(X) = (1/2)[v(X) - iv(iX)], \qquad s(X) = (1/2)[v(X) + iv(iX)]$$

define mutually commuting complex linear and anti-linear representations, with sum $r+s=v$. In the present situation, the operators are only partially defined, but the same relations hold relative to the usual calculus for such operators, and all of the operators in question are defined on and leave invariant the domain $\mathfrak{D}$.

The finite representation R may be obtained by first defining $R_0(a)$ for a near e, on the domain $\mathfrak{D}$, by the equation

$$R_0(e^X)z = \sum_n \frac{(r(X))^n}{n!} z.$$

Then $R_0(a)$ is defined, and adjoint to the densely-defined operator on $\mathfrak{D}$,

$$z \to \sum_n \frac{(r(X)^*)^n}{n!} z.$$

Therefore its closure exists and may be designated $R(a)$. Furthermore, $R(e^X)z$ is an analytic function of X, with values in $\mathfrak{H}$, in fact it should be noted that if

$$F(X) = \sum_n \frac{(v(X))^n}{n!}$$

(on the domain of all vectors for which the series is convergent in the fashion indicated above), then $F(X)$ may be extended by analyticity to complex values of X near $X=0$. In addition, by the absolute convergence of the series involved, if Y and Y' are commuting elements of the complexification of $\mathcal{G}$ which are sufficiently close to 0, then $F(Y+Y')z=F(Y)F(Y')z$. This is relevant since $R(e^X)z$ may be expressed as $F((X-i'X)/2)z$, $z\in\mathfrak{D}$, where i' gives the action of the complex unit on the direct sum of $\mathcal{G}$ with itself (and must be distinguished from the action in $\mathcal{G}$ giving the original complex structure in it).

The same holds when adjoints are taken, and it follows that the contragredient representation to R exists and is of the appropriate form for R to be an analytic representation with domain $\mathfrak{D}$. By symmetry, S is such also, and the relations given in Theorem 1 involving R and S jointly follow by the same argument concerning products of absolutely convergent series. The converse then follows straightforwardly by similar arguments in reverse order.

The important case when V is unitary has additional features as described in

THEOREM 2. *Any continuous unitary representation U on a Hilbert space of a complex connected semi-simple Lie group is locally of the form*

$$U(a) = R(a)R(a^{-1})^*,$$

*where R is a holomorphic representation by normal operators, which commutes with its contragredient representation, and whose maximal infinitesimal representation r is also normal. A closed linear subspace is invariant under U if and only if it is *-invariant under r.*

By [2], U is analytic on the domain $\mathfrak{D}$ of all "analytic vectors," and only the normality of R and r require proof. Now if $u(X)$ is for any X in $\mathcal{G}$ the skew-adjoint generator of the one-parameter group $\{U(e^{tX}): t \text{ real}\}$, then $r'(X)=(1/2)[u(X)-iu(iX)]$ is normal, since it has the form $A+iB$ for skew-adjoint operators A and B which commute in the strong sense that their spectral projections do so (which is the case since the one-parameter groups generated by $u(X)$ and $u(iX)$ commute, by the commutativity of X with iX). Furthermore, $r'(X)$ has $\mathfrak{D}$ as a domain of control by virtue of the fact that any such normal operator $A+iB$ with A and B skew-adjoint is the closure of its restriction to any domain contained in the domain of A^2+B^2 and on which the latter operator has an essentially self-adjoint restriction; this fact is applicable by virtue of Nelson's results [2] which imply that A^2+B^2 is here defined and essentially self-adjoint on $\mathfrak{D}$. To establish this fact, let C be the positive self-adjoint square root of $-(A^2+B^2)$, let $\mathfrak{K}$ be the graph of the mapping $y\to(y, Ay, By)$ from the domain of C to the three-fold direct sum of the Hilbert space with itself, and impose on $\mathfrak{K}$ the inner product

$$\langle (y, Ay, By), (y', Ay', By')\rangle = (y, y') - (Ay, Ay') - (By, By'),$$

relative to which $\mathfrak{K}$ is a Hilbert space. If the image $\mathfrak{K}_0$ of the given domain $\mathfrak{D}$ under the indicated mapping were not dense, there would exist an orthogonal element to $\mathfrak{K}_0$ in $\mathfrak{K}$, i.e. an element (y_0, Ay_0, By_0) such that

$$(y, y_0) - (Ay, Ay_0) - (By, By_0) = 0, \qquad y \in \mathfrak{D}.$$

But this implies that $((I-C)y, y_0)=0$, while for any nonnegative essentially self-adjoint operator C', the range of $I-C'$ is dense, so that y_0 must vanish. This means that every element y in the domain of $A+iB$ may be approximated by elements of $\mathfrak{D}$ in such a fashion that the action of A and B are simultaneously approximated, which is equivalent to the cited fact.

Now $R(e^X)$ agrees on $\mathfrak{D}$ with $e^{r(X)}$, as follows from the uniqueness of the solution $u(t)$ of the differential equation $u'(t)=Nu(t)$, $u(0)=z$, N normal, which both operators may be regarded as giving a solution of, evaluated at $t=1$. (Note that if w is any vector in the domain of e^{tN}, $0\leqq t\leqq 1$, then $(d/dt)e^{tN}w$ exists and equals $Ne^{tN}w$, as follows directly by diagonalizing N.) The uniqueness in question may be reduced to that for the case when N is bounded by multiplication of both sides of the equation by arbitrary bounded spectral projections for N. Thus $R(e^X)\subset e^{r(X)}$; on the other hand, the same relation holds also for the contragredient representation S, $S(e^X)=R(e^{-X})^*$, so that replacing X by $-X$, $R(e^X)^*\subset e^{r(X)*}$. Taking adjoints in the last relation, it results that $e^{r(X)}\subset R(e^X)$, so $R(e^X)=e^{r(X)}$, showing that $R(a)$ is normal, and has $\mathfrak{D}$ as a domain of control.

The implications of Theorem 2 for the representations of compact groups may be stated as

THEOREM 3. *A compact semi-simple Lie group has a complete set of *-irreducible analytic local representations by normal operators in Hilbert space commuting with their respective contragredients. Conversely, any such representation V is the restriction to the compact subgroup of the holomorphic constituent of an irreducible continuous representation U of the corresponding complex group by unitary operators in Hilbert space, whose maximal infinitesimal form is given in terms of that for V by the equation*

$$u(X + iY) = \text{closure of } v(X) - v(X)^* + i[v(Y) + v(Y)^*].$$

Since the infinitesimal representation r obtained in the proof of Theorem 2 is complex linear, it is determined by its values on a compact subalgebra, which provide an infinitesimal representation v which has a finite form V of the type described. It is straightforward to verify that $u(X+iY)$ agrees on $\mathfrak{D}$ with the indicated operator, and so has the indicated form. Since the connected complex group has a complete set of irreducible continuous unitary representations U, the compact form has a complete set of *-irreducible representations V of the type described.

We add several comments as follows. Since an open simple Lie group has no finite-dimensional unitary representations, all of the representations V described in Theorem 3 except for the unitary ones are infinite-dimensional, in the case of a simple group. The map $(a, b)\rightarrow V(a)V(b^{-1})^*$ provides a fully irreducible analytic local representation of the direct product of the compact group with itself. In the case of a real semi-simple group, any continuous unitary representation in a Hilbert space gives rise to a pair of commuting analytic (but not necessarily normal or mutually contragredient) representations of the compact form, by virtue of the present Theorem 1 and Theorem 1 of [1], from which the original representation may be reconstructed. The algebraic decomposition $u=r+s$ used in the proof of Theorem 1 has a formal analogy with devices implicit in the literature, and notably with the employment in quantum mechanics of the creation and annihilation operators in place of the hermitian canonical variables.

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