RESEARCH ANNOUNCEMENTS

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ADDITIVITY OF THE GENUS OF A GRAPH

BY JOSEPH BATTLE, FRANK HARARY,[1] YUKIHIRO KODAMA,[1] AND J. W. T. YOUNGS[1]

Communicated June 26, 1962

In this note a graph G is a finite 1-complex, and an imbedding of G in an orientable 2-manifold M is a geometric realization of G in M. The letter G will also be used to designate the set in M which is the realization of G. Manifolds will always be orientable 2-manifolds, and $\gamma(M)$ will stand for the genus of M. Given a graph G the *genus* $\gamma(G)$ of G is the smallest number $\gamma(M)$, for M in the collection of manifolds in which G can be imbedded.

A *block* of G is a subgraph B of G maximal with respect to the property that removing any single vertex of B does not disconnect B. (A block with more than two vertices is a "true cyclic element" in Whyburn [3].) Given G there is a unique finite collection $\mathfrak{B}$ of blocks B of G such that $G=\bigcup B$, $B\in\mathfrak{B}$. The collection $\mathfrak{B}$ is called the *block decomposition* of G. If G is connected and $\mathfrak{B}$ contains k blocks; then they may be listed in an order $B_1, \cdots, B_k$ such that

$$(1)\qquad \bigcup_1^j B_i \text{ is connected, and } B_{j+1}\cap\bigcup_1^j B_i \text{ is a vertex of } G \quad \text{for } j=1, \cdots, (k-1).$$

A *2-cell imbedding* of G is an imbedding in a manifold M such that each component of $(M-G)$ is an open 2-cell. (See Youngs [4]). The *regional number* $\delta(G)$ of a graph G is the maximum number of components of $(M-G)$ for all possible 2-cell imbeddings of G. In [4] it was shown that if G is connected then

$$(2)\qquad \delta(G) = 2 - \chi(G) - 2\gamma(G)$$

where $\chi(G)$ is the Euler characteristic of G.

The object of this note is to prove two formulas about the block decomposition of a connected graph G with k blocks $B_1, \cdots, B_k$:

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[1] Partial support for this research was provided by the U. S. Naval Research Laboratory and the National Science Foundation.

$$\gamma(G) = \sum_{1}^{k} \gamma(B_i). \tag{I}$$

$$\delta(G) = 1 - k + \sum_{1}^{k} \delta(B_i). \tag{II}$$

Whereas equation (I) is intuitively expected, it is by no means a triviality; there is a great deal below the surface. Moreover, it has important applications to be made elsewhere. The proof uses two lemmas.

LEMMA 1. *If G_1, G_2 and G are connected graphs such that $G=G_1\cup G_2$ and $G_1\cap G_2=v$ (a vertex of G), then $\gamma(G)\leqq\gamma(G_1)+\gamma(G_2)$.*

PROOF. Let each G_i be imbedded in an orientable 2-manifold M_i such that

$$\gamma(M_i) = \gamma(G_i), \qquad i = 1, 2. \tag{3}$$

For convenience, let the vertex v be designated by v_i when it is considered as a vertex of G_i, $i=1, 2$. There is an open 2-cell C_i in M_i with a simple closed boundary curve J_i such that $(C_i\cup J_i)\cap G_i=v_i$. Identify J_1 of (M_1-C_1) with J_2 of (M_2-C_2) so that v_1 is identified with v_2. This provides a closed 2-manifold $(M_1-C_1)\cup(M_2-C_2)$ containing G, hence

$$\gamma(G) \leqq \gamma(M). \tag{4}$$

On the other hand, an easy computation involving the Euler characteristic shows that

$$\gamma(M) = \gamma(M_1) + \gamma(M_2). \tag{5}$$

The lemma follows immediately from (3), (4) and (5).

LEMMA 2. *If G is a connected graph having a subgraph G_1 and a block G_2 such that $G=G_1\cup G_2$, and $G_1\cap G_2=v$ (a vertex of G), then $\gamma(G)\geqq\gamma(G_1)+\gamma(G_2)$.*

PROOF. First, note that under these hypotheses, G_1 is a connected subgraph of G. Consider an imbedding of G in an orientable 2-manifold M such that

$$\gamma(M) = \gamma(G). \tag{6}$$

Since G_2 is a block, (G_2-v) is connected. Hence G_2-v lies in a component S of $M-G_1$. Using the techniques of [4, §3], take a triangulation τ of M such that G is a subcomplex of the 1-skeleton of τ and let τ_2 be the second barycentric subdivision of τ. Dealing entirely with τ_2, consider the open star R of $\mathrm{Fr}(S)$, the frontier of S. The intersection of $\mathrm{Fr}(R)$ and S will be s 1-spheres $J_1, \cdots, J_s$ where $s \geqq$ the number of components of $\mathrm{Fr}(S)$. Moreover, if Q is the open star of $\bigcup J_i$, relative to τ_2, then the components of $Q \cap R$ are open cylinders $L_1, \cdots, L_s$, and $\mathrm{Fr}(L_i)$ has two components, J_i and a subset of G_1, $i=1, \cdots, s$. In view of these facts and because G_1 is connected, the set $M-\bigcup J_i$ has two components, S_1 and S_2, where the notation is chosen so that $S_1 \supset G_1$ and $S_2 = S - \bar{S}_1$. If $N_i = \bar{S}_i$, then N_i is an orientable 2-manifold with boundary curves $J_1, \cdots, J_s$.

On capping the boundary curves $J_1, \cdots, J_s$ of N_1 with 2-cells $C_1, \cdots, C_s$ respectively, one obtains an orientable 2-manifold $M_1 = N_1 \cup \bigcup C_i$. Note that $G_1 \subset M_1$, hence

$$\gamma(G_1) \leqq \gamma(M_1). \tag{7}$$

Suppose P is an orientable 2-manifold with boundary curves $K_1, \cdots, K_s$, obtained by removing s open 2-cells from a 2-sphere. Select orientations on N_2 and P. These selections will induce orientations on $J_1, \cdots, J_s$ and $K_1, \cdots, K_s$. Identify J_i with K_i so that the orientations match for $i=1, \cdots, m$. This produces an orientable 2-manifold $M_2 = N_2 \cup P$. Resorting once more to the Euler characteristic (compare Ringel [1, pp. 56–57]), one finds

$$\gamma(M) = \gamma(M_1) + \gamma(M_2). \tag{8}$$

Now consider that part of G_2 which lies in $N_2 \subset M_2$. Since τ_2 is a second barycentric subdivision, each arc of G_2 incident on v cuts precisely one boundary curve J_i of N_2 and cuts it exactly once. Take any point v_0 in $P - \bigcup K_i$. Then it is possible to join v_0 with each point of $G_2 \cap \bigcup J_i$ by arcs in P such that any pair of these arcs intersect only at v_0. These arcs, together with $G_2 \cap N_2$, provide an imbedding of G_2 in M_2, hence

$$\gamma(G_2) \leqq \gamma(M_2). \tag{9}$$

The lemma follows from (6), (7), (8) and (9).

THEOREM 1. *If G is a connected graph having k blocks $B_1, \cdots, B_k$, then $\gamma(G) = \sum_1^k \gamma(B_i)$.*

The result is obtained by a straightforward induction using (1) and both lemmas.

If a graph G is *not* connected suppose it has n components. Clearly there is a connected graph $H \supset G$ such that H has one vertex and n arcs not in G. Each of these n arcs is a block of H, and a block with genus zero. Consequently, the following statements are true:

COROLLARY 1. *The genus of any graph (connected or not) is the sum of the genuses of its blocks.*

COROLLARY 2. *The genus of a graph is the sum of the genuses of its components.*

THEOREM 2. *If G is a connected graph having k blocks $B_1, \cdots, B_k$, then $\delta(G) = 1 - k + \sum_1^k \delta(B_i)$.*

PROOF. Because of (1),

$$\chi(G) = \sum_1^k \chi(B_i) - (k - 1).$$

Hence by (2),

$$\delta(G) = 2 - \sum_1^k \chi(B_i) + (k - 1) - 2\gamma(G).$$

Since each block is connected, use (2) again to obtain

$$\begin{aligned}\delta(G) &= (1 + k) - \sum_1^k [2 - \delta(B_i) - 2\gamma(B_i)] - 2\gamma(G) \\ &= (1 - k) + \sum_1^k \delta(B_i) - 2\left[\gamma(G) - \sum_1^k \gamma(B_i)\right].\end{aligned}$$

The result now follows from Theorem 1 which states that the last term vanishes.

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