SINGULAR PERTURBATIONS

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Let P_ϵ designate the problem of finding a solution of the differential equation

$$\epsilon y'' + F(t, y, y', \epsilon) = 0, \qquad 0 \leqq t \leqq 1, \tag{1}$$

that satisfies the boundary conditions

$$y(0) = \alpha(\epsilon), \qquad y(1) = \beta(\epsilon). \tag{2}$$

Here ϵ is a small positive parameter approaching zero. We envisage circumstances under which $y = y(t, \epsilon)$ approaches a limit nonuniformly in t as $\epsilon \to 0+$, the nonuniformity occurring at $t = 0$. Accordingly, the limiting problem P_0 involves the differential equation

$$F(t, u, u', 0) = 0, \qquad 0 \leqq t \leqq 1, \tag{3}$$

with the single boundary condition

$$u(1) = \beta(0). \tag{4}$$

Partial derivatives will be denoted by subscripts, thus $F_y = \partial F/\partial y$, etc.

For a solution $u = u(t)$ of (3) we define the function ϕ and the region D_δ by

$$\phi(t) = \int_0^t F_{y'}(\tau, u(\tau), u'(\tau), 0)d\tau,$$

$$D_\delta = [(t, y, y', \epsilon): 0 \leqq t \leqq 1, \ | y - u(t) | < \delta,$$
$$| y' - u'(t) | < \delta(1 + \epsilon^{-1}e^{-\phi(t)/\epsilon}), \ 0 < \epsilon < \epsilon_0].$$

ASSUMPTIONS. (A) *The problem* P_0, (3) *and* (4), *possesses a solution* u *which is twice continuously differentiable on* $[0, 1]$.

(B) *For some* $\delta > 0$, F *possesses partial derivatives of the first and second orders with respect to* y *and* y' *in* D_δ, *and* F *as well as these partial derivatives are continuous functions of* t, y, y' (*for fixed* ϵ).

(C) $F(t, u(t), u'(t), \epsilon) = O(\epsilon)$; $q(t, \epsilon) = F_y(t, u(t), u'(t), \epsilon) = O(1)$; $p(t, \epsilon) = F_{y'}(t, u(t), u'(t), \epsilon) = \phi'(t) + \epsilon p_1(t, \epsilon)$ *where* ϕ *is twice continu-*

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ously differentiable in $[0, 1]$, $\phi(0)=0$, $\phi'(t)>0$, p_1 *is a continuous function of t and* $p_1(t, \epsilon)=O(1)$; $F_{yy}(t, y, y', \epsilon)=O(1)$, $F_{yy'}(t, y, y', \epsilon)=O(1)$, $F_{y'y'}(t, y, y', \epsilon)=O(\epsilon)$. *All order relations (here and later) hold as* $\epsilon \to 0+$, *uniformly in all the other variables in their respective ranges.*

(D) $\beta(\epsilon)-\beta(0)=O(\epsilon)$.

(E) $F_y(t, y, y', \epsilon)=O(1)$, $F_{y'}(t, y, y', \epsilon)\geqq B>0$ *in* D_δ.

Coddington and Levinson [1] discussed the problem P_ϵ under the assumption that F is linear in y' (and, for the sake of simplicity, F, α, and β are independent of ϵ). Their differentiability conditions on F were somewhat milder than ours, and they proved that $y(t, \epsilon) \to u(t)$ and $y'(t, \epsilon)\to u'(t)$ as $\epsilon\to 0+$, uniformly on any interval $0<\delta\leqq t\leqq 1$. Wasow [2] also assumed F to be linear in y' and in addition he assumed F to be analytic in y and ϵ. He established results, including asymptotic expansions, valid throughout $[0, 1]$. Briš [3] considered F that are nonlinear in y' and independent of ϵ: his conditions neither include ours, nor are they included by ours.

THEOREM. *Under assumptions* (A) *to* (D) *there exists a* $\mu_0>0$, *independent of* ϵ, *so that whenever* $|\alpha(\epsilon)-u(0)|<\mu_0$, *the boundary value problem* P_ϵ *possesses a solution* $y=y(t, \epsilon)$ *for each sufficiently small* ϵ. *Moreover,* $y=u+v+w$, *where* $y^*=u+v$ *satisfies* (1) *and the second boundary condition* (2), $v(t)$, $v'(t)=O(\epsilon)$, *and* $w(t)$, $\epsilon w'(t)=O(\exp[-\phi(t)/\epsilon])$. *Under the further assumption* (E), y *is the only solution of* P_ϵ *in* D_δ.

The proof of this result follows the pattern set by Wasow in that first v is constructed by "linearizing" the differential equation "around u," and then w is constructed by linearizing around $y^*=u+v$. The technique employed in carrying out these steps differs from Wasow's. While Wasow employs power series—asymptotic expansions combined with integral equations in the case of v, and convergent power series expansions in the case of w, all constructions here are based directly on integral equations. This procedure makes the analyticity assumptions on F unnecessary and makes it easier to include differential equations which are not linear in y'.

First the linear differential equation

$$\epsilon V'' + p(t, \epsilon)V' + q(t, \epsilon)V = 0 \tag{5}$$

is investigated: p and q are subject to assumptions obtained from (B) and (C) if one sets $F=py'+qy$. The results are known in essence from the asymptotic theory of linear differential equations. They are obtained here, under milder differentiability conditions than is usual, from the integral equation

$$(6) \quad V(t) = 1 + \int_0^1 (1 - e^{\theta(\tau,1)})\overline{V}(\tau)d\tau - \int_0^t (1 - e^{\theta(\tau,t)})\overline{V}(\tau)d\tau,$$

where

$$\theta(s, t) = [\phi(s) - \phi(t)]/\epsilon,$$
$$\overline{V}(t)\phi'(t) = [p(t, \epsilon) - \phi'(t) + \epsilon\phi''(t)/\phi'(t)]V'(t) + q(t, \epsilon)V(t).$$

The principal results on (6) state the existence of two solutions V_1 and V_2 such that

$$(7) \quad \begin{aligned} &V_1(1) = 1, \qquad V_1'(0) = 0, \qquad V_1(t) = e^{\psi(1)-\psi(t)} + O(\epsilon), \\ &V_1'(t) = -\psi'(t)e^{\psi(1)-\psi(t)}\left[1 - e^{-\phi(t)/\epsilon}\right] + o(1), \end{aligned}$$

$$(8) \quad \begin{aligned} &V_2(0) = 1, \qquad V_2(t) = e^{\psi_2(t)-\phi(t)/\epsilon} + O(\epsilon e^{-\phi(t)/\epsilon}), \\ &V_2'(t) = -\epsilon^{-1}\phi'(t)e^{\psi_2(t)-\phi(t)/\epsilon} + O(e^{-\phi(t)/\epsilon}), \end{aligned}$$

where

$$\psi(t) = \psi(t, \epsilon) = \int_0^t \frac{q(\tau, \epsilon)}{\phi'(\tau)} d\tau,$$

$$\psi_2(t, \epsilon) = \psi(t, \epsilon) - \int_0^t P_2(t, \epsilon)dt.$$

V_1 is the solution of (6), and V_2 is obtained by a transformation. Asymptotic forms of other solutions of (5) follow readily from (7) and (8).

We now set $y^* = u + v$ in (1) and rewrite this differential equation in the form

$$(9) \quad \epsilon v'' + p(t, \epsilon)v' + q(t, \epsilon)v = G(t, v, v', \epsilon),$$

where p and q are as in assumption (C). A solution of (9) satisfying $v(1) = \beta(\epsilon) - \beta(0)$ is obtained as a solution, by successive approximations, of the integral equation

$$(10) \quad \begin{aligned} v(t) = &\left[\beta(\epsilon) - \beta(0) - \int_0^1 K(1, s)G(s, v(s), v'(s), \epsilon)ds\right] V_1(t) \\ &+ \int_0^t K(t, s)G(s, v(s), v'(s), \epsilon)ds, \end{aligned}$$

where

$$K(t, s) = \frac{1}{\epsilon} \frac{V_1(s)V_2(t) - V_1(t)V_2(s)}{V_1(s)V_2'(s) - V_1'(s)V_2(s)};$$

and this solution is shown to possess the properties asserted in the theorem.

Setting $y = y^* + w = u + v + w$ in (1), we see that the boundary layer correction w satisfies a differential equation that can be written as

$$\epsilon w'' + p^*(t, \epsilon)w' + q^*(t, \epsilon)w = G^*(t, w, w', \epsilon), \tag{11}$$

where

$$p^*(t, \epsilon) = F_{y'}(t, y^*(t), y^{*\prime}(t), \epsilon), \quad q^*(t, \epsilon) = F_y(t, y^*(t), y^{*\prime}(t), \epsilon).$$

w also satisfies the boundary conditions

$$w(0) = \alpha(\epsilon) - u(0) - v(0) = \mu, w(1) = 0.$$

The linear part of (11) is again of the form (5), with p and q replaced by p^* and q^*. Constructing K^* as before, and setting

$$V_3^*(t) = \frac{V_1^*(t)V_2^*(1) - V_1^*(1)V_2^*(t)}{V_1^*(0)V_2^*(1) - V_1^*(1)V_2^*(0)},$$

we see that w satisfies the integral equation

$$\begin{aligned} w(t) = {} & \left[\mu + \int_0^1 K^*(0, s)G^*(s, w(s), w'(s), \epsilon)ds\right] V_3^*(t) \\ & - \int_t^1 K^*(t, s)G^*(s, w(s), w'(s), \epsilon)ds. \end{aligned} \tag{12}$$

Again the integral equation can be solved by successive approximations and those properties of w asserted in the theorem follow.

Lastly, the uniqueness of the solution of P_ϵ is established as follows. That solution, $y = y(t, \epsilon, \gamma)$ of (1) satisfying the initial conditions

$$y(0) = \alpha(\epsilon), \qquad y'(0) = \gamma$$

is unique as far as it remains in D_δ, and this solution is differentiable with respect to γ. $z = \partial y/\partial\gamma$ satisfies the "variational equation"

$$\epsilon z'' + F_{y'}(t, y(t), y'(t), \epsilon)z' + F_y(t, y(t), y'(t), \epsilon)z = 0 \tag{13}$$

and the initial conditions $z(0) = 0$, $z'(0) = 1$. Now, (13) is again of the form (5) and it can be shown that $z(1) > 0$. Thus, $y(1, \epsilon, \gamma)$ is a strictly increasing function of γ, and for a fixed sufficiently small ϵ there is at most one $\gamma=\gamma(\epsilon)$ so that $y(1, \epsilon, \gamma(\epsilon))=\beta(\epsilon)$.

A detailed presentation of this result together with some further developments will appear elsewhere.

REFERENCES

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