EXAMPLES OF PERIODIC MAPS ON EUCLIDEAN SPACES WITHOUT FIXED POINTS

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Let T be a map of period r on a Euclidean space E^n. Smith seems to have been the first to consider fixed points of T. He showed that T has a fixed point if r is a prime in [4], extended this result to r a power of a prime, and raised the question concerning the existence of a fixed point for r not a prime power in [5]; also cf. Problem 33 in [3]. Conner and Floyd gave an example of a contractible manifold M_r for every r not a prime power, and a map T of period r on M_r without fixed points [2]. They conjectured that M_r was a Euclidean space. This note shows that a slight modification of their example is Euclidean, hence:

THEOREM. *If r is an integer which is not a power of a prime, then there exists a triangulation τ of E^{9r}, a map T of period r on E^{9r} without fixed points, and T is simplicial relative to τ.*

I wish to express my indebtedness to Professor Floyd for his help and encouragement.

Preliminaries. Let K be a subcomplex of a Euclidean space E under a triangulation σ. Let $\sigma_K^{(1)}$ be the subdivision of σ obtained by adding barycenters of all simplexes not contained in K, cf. [6, p. 251]. $\sigma_K^{(i+1)} \equiv (\sigma_{iK}^{(i)})_K^{(i)}$. If K is the empty complex, $\sigma_K^{(i)} \equiv \sigma^{(i)}$, the usual ith barycentric subdivision. Denote the closed star of K in σ by $V(K, \sigma)$ and let $V^2(K, \sigma) = V(V(K, \sigma), \sigma)$. $N_W(K, \sigma) \equiv V(K, \sigma_K^{(2)})$ is a "regular" neighborhood of K; cf. [6, p. 293]. If K is a contractible finite subcomplex having dimension m and $E = E^n$, where $n \geqq 2m+5$, then it follows from Corollary 3 in [6, p. 298] that $N_W(K, \sigma)$ is an n-cell. Much use is made of this fact; however it will be convenient later to use the following neighborhood: $N_1(K, \sigma) \equiv V(K^{(2)}, \sigma^{(2)})$, i.e. the star of K (subdivided twice barycentrically) in $\sigma^{(2)}$. Since it will be necessary to use Whitehead's result, but only in a topological way (i.e. noncombinatorial), it suffices to show that $N_W(K, \sigma)$ and $N_1(K, \sigma)$ are homeomorphic. This can be done by looking at an n-simplex ρ in the triangulation $\sigma_K^{(1)}$ which intersects K, and constructing a canonical homeomorphism of $N_W(K, \sigma) \cap \rho$ and $N_1(K, \sigma) \cap \rho$ in such a way that two such homeomorphisms match on p-faces, $p < n$. Let $\rho = \rho_0 \circ \rho_1$,

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the join, where ρ_0 and ρ_1 are simplexes in $\sigma_K^{(1)}$ with $\rho_0 \cap K = \emptyset$ and $\rho_1 \subseteq K$. It is easy to show that every segment $A_{x_0,x_1} = \{x_t = (1-t)x_0 + tx_1 | t \text{ in } [0, 1]\}$ where x_0 is in ρ_0, x_1 in ρ_1, intersects $N_W(K, \sigma)$ and $N_1(K, \sigma)$ in a desirable way; namely there exist t_W and t_1 in (0, 1) such that $A_{x_0,x_1} \cap N_W(K, \sigma) = \{x_t | t_W \leq t \leq 1\} \equiv A_W$ and $A_{x_0,x_1} \cap N_1(K, \sigma) = \{x_t | t_1 \leq t \leq 1\} \equiv A_1$. Now map A_W onto A_1 linearly and do this for each pair x_0, x_1 in ρ_0 and ρ_1. The regular neighborhoods used here will now be N_1, and we define $N_{i+1}(K, \sigma) \equiv N_i(N_1(K, \sigma), \sigma^{(2)})$, a subcomplex in E under $\sigma^{(2i+2)}$. Clearly if K_1 is a subcomplex under σ_1, $K_1 \subseteq K$, and σ_1 refines σ, then $N_i(K_1, \sigma_1) \subseteq N_i(K, \sigma)$, for all i.

For a given r in the theorem let K be the Conner-Floyd example of a 4-dimensional, star-finite, contractible complex with Z_r acting without fixed points [2, p. 360]. K is the union of simplicial mapping cylinders C_i whose "beginning" B_i and "end" E_i are 3-spheres. C_i is the simplicial analogy of the ordinary mapping cylinder defined, in this case, by a simplicial map f^* of B_i into E_i which is inessential. Also $E_i = B_{i+1}$. For a more exact description the reader is referred to [2]. It will also be assumed that K is imbedded as a subcomplex of $E = E^{9r}$ with triangulation σ, S is a simplicial map of E onto itself of period r (in fact, $S(x_1, \cdots, x_r) = (x_2, \cdots, x_r, x_1)$ for $x_i \in E^9$), K is an invariant set under S, and $S|K$ is the generator of Z_r. This, too, was done in [2]. σ can be taken fine enough so that $V(C_i, \sigma)$ and $V(C_j, \sigma)$ are disjoint if $|i-j| > 1$. Since S is simplicial and K, invariant, $N_i(K, \sigma)$ is invariant for each i, hence $E' \equiv \bigcup_{i=1}^{\infty} N_i(K, \sigma)$ is invariant. Since S has no fixed points in K and $E' \subseteq \text{Int } V(K, \sigma)$, S has no fixed points in E'. It will be shown that E' is Euclidean. To do this it will suffice to express E' as the union of cubes $\{I_j\}_{j=1}^{\infty}$ with $I_j \subseteq \text{Int } I_{j+1}$ and then one could use a recently announced result of M. Brown.[2] However, in this special case, where each I_j is in E^n, $\bigcup_{j=1}^{\infty} I_j$ is seen to be Euclidean by an easy application of another result of Brown which is in the literature [1]. For by the characterization of a tame S^{n-1} in E^n given there, and by taking I_j to be a slightly smaller concentric cube, there is no loss of generality in assuming that Bd I_j is a tame S^{n-1} in E^n for each j. Hence the complement J of $\bigcup_{j=1}^{\infty} I_j$ in the one point compactification of E^n is the intersection of decreasing cubes, i.e. cellular [1], hence $E^n \approx S^n - J = (E^n \cup \infty) - J = \bigcup_{j=1}^{\infty} I_j$.

LEMMA. *Given any positive integer i there exists a subdivision σ_i of σ and a finite contractible complex K_i in σ_i such that*

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[2] *The monotone union of open n-cells is an open n-cell*, Notices Amer. Math. Soc. vol. 7 (1960) p. 478.

(1) $L_i \equiv \bigcup_{j \leq i} C_j \subseteq K_i$.
(2) *σ_i agrees with σ on $V(L_i, \sigma)$, hence $N_j(L_i, \sigma) \subseteq N_j(K_i, \sigma_i)$, for all j.*
(3) *$N_j(K_i, \sigma_i) \subseteq N_j(L_{i+2}, \sigma)$, for all j.*

PROOF. Note that if D is a 4-cell and h is a homeomorphism from the Bd D onto the end E_{i+1} of C_{i+1}, then since E_{i+1} is a strong deformation retract of C_{i+1}, hence of L_{i+1}, it follows that the identification space $L_{i+1} \cup D/h$ is contractible. The proof of the lemma depends on getting a simplicial representation K_i of this identification space close to L_{i+2}.

First we produce a map f of D into C_{i+2}. For a simplicial model of D take the cone over the simplicial 3-sphere B, where B is a copy of B_{i+2}, the beginning of C_{i+2}. $D = \{(b, t) \mid b \in B, t \in I\}/B \times 1$. Let $D' \equiv \{(b, t) \in D \mid t \in [0, 1/2]\}$, $D'' \equiv \{(b, t) \in D \mid t \in [1/2, 1]\}$. $D' \cap D'' \equiv B'$. Now map D' into C_{i+2} by f' such that $f' \mid B: B \to B_{i+2}$ is a simplicial isomorphism, $f'(B') \subseteq E_{i+2}$ and $f' \mid B'$ is inessential (it can be taken to be, essentially, f^*). Hence there is a map $f'': D'' \to E_{i+2}$ such that $f'' \mid B' = f' \mid B'$. Then $f \mid D' = f'$ and $f \mid D'' = f''$ define f.

Let ϵ be so small that the ϵ-neighborhood (under the usual metric for E) of C_{i+2} is contained in $N_1(C_{i+2}, \sigma)$. Since dim $E = 9r > 8$ we can get g, an ϵ-approximation of f, which imbeds D in E (hence in $N_1(C_{i+2}, \sigma)$) and such that

(a) $g \mid B = f \mid B$ and $g(D) \cap L_{i+1} = B_{i+2} = g(B)$,

(b) g maps linearly (using the vector space structure of E) each simplex in the kth barycentric subdivision of D, for some k.

The usual technique is used, that of subdividing D so that images of simplexes under f are small relative to ϵ, then choosing a point near each image of a vertex (keeping fixed images of vertices in B) so that the set of all such points is in general position, and then extending the obvious vertex map linearly.

Now we get a subdivision of $V(C_{i+2}, \sigma)$ so that $g(D)$ may be regarded as a subcomplex. One way of getting this would be to regard each 4-simplex in $g(D)$ as a subsimplex of a rectilinear n-simplex in E. Consider the $(n-1)$-planes determined by the $(n-1)$-faces of such n-simplexes, one chosen for each 4-simplex in $g(D)$. The triangulation σ, together with this finite collection of $(n-1)$-planes, partitions $V(C_{i+2}, \sigma)$ into convex polyhedral sets which can then be triangulated. Furthermore this triangulation σ_i can be taken so fine that $N_1(g(D), \sigma_i) \subseteq N_1(C_{i+2}, \sigma)$. Now extend the triangulation σ_1 to all of E keeping σ on $F = \mathrm{Cl}(E - V^2(C_{i+2}, \sigma))$. This can be done by triangulating the "ring" $R = \mathrm{Cl}(V^2(C_{i+2}, \sigma) - V(C_{i+2}, \sigma))$ without introducing any new vertices in $V^2(C_{i+2}, \sigma) - V(C_{i+2}, \sigma)$. Each simplex under σ in R, say ρ, may be regarded as a join of two simplexes ρ_1 and ρ_2 in F and

$V(C_{i+2}, \sigma)$ respectively, where ρ_2 has been subdivided under σ_i. Triangulate ρ by taking the joins of ρ_1 with the small simplexes in ρ_2 under σ_i. Do this for each ρ in R getting a triangulation σ_i of E which is a subdivision of σ, which agrees with σ on F. Condition (2) of the lemma follows from $V(L_i, \sigma) \subseteq F$.

Define $K_i = L_{i+1} \cup g(D)$, a contractible subcomplex under the triangulation σ_i. Condition (1) is clearly satisfied. Since σ_i refines σ, $N_j(L_{i+1}, \sigma_i) \subseteq N_j(L_{i+1}, \sigma)$ and since $N_1(g(D), \sigma_i) \subseteq N_1(C_{i+2}, \sigma)$, $N_j(g(D), \sigma_i) \subseteq N_j(C_{i+2}), \sigma)$ and it follows that $N_j(K_i, \sigma_i) \subseteq N_j(L_{i+1}, \sigma) \cup N_j(g(D), \sigma_i) \subseteq N_j(L_{i+1}, \sigma) \cup N_j(C_{i+2}, \sigma) = N_j(L_{i+2}, \sigma)$, hence condition (3) is satisfied and the lemma proved.

Proof of the theorem. Using the notation of the lemma, $N_1(K_i, \sigma_i) \approx N_W(K_i, \sigma_i)$ is an n-cell by [6], hence $N_i(K_i, \sigma_i)$, which can be expressed as $N_1(N_{i-1}(K_i, \sigma_i), \sigma_i^{(2i-2)}) \approx N_W(N_{i-1}(K_i, \sigma_i), \sigma_i^{(2i-2)})$, is an n-cell, which we designate by I_i. Using the lemma we get:

$$N_1(L_1, \sigma) \subseteq I_1 \subseteq N_1(L_3, \sigma) \subseteq N_3(L_3, \sigma) \subseteq I_3 \subseteq N_3(L_5, \sigma) \subseteq N_5(L_5, \sigma) \subseteq I_5 \subseteq \cdots$$

and I_{2i-1} is contained in the interior of I_{2i+1}. Then $E' = \bigcup_{i=1}^{\infty} N_i(K, \sigma) = \bigcup_{i=1}^{\infty} N_i(L_i, \sigma) = \bigcup_{j=1}^{\infty} I_{2j-1}$ is Euclidean. The map T is, of course, $S | E'$, and the invariant triangulation is obtained in the following way. $N_i(K, \sigma)$ is a complex in $\sigma^{(2i)}$, and it is subdivided twice without subdividing $N_{i-1}(K, \sigma)$, i.e. $N_i(K, \sigma)$ becomes a complex in $(\sigma^{(2i)})^2_{N_{i-1}(K,\sigma)^{(2)}}$.

Added in proof. D. R. McMillan has communicated to me an alternate (and simpler) way of producing examples in E^{18r} making use of recent results of his in *Cartesian products of contractible open manifolds*, Bull. Amer. Math. Soc., this issue, pp. 510-514

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