CLOSED IDEALS IN GROUP ALGEBRAS

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Let $A(G)$ be the set of all Fourier transforms on the locally compact abelian group G, i.e., the set of all f of the form

$$f(x) = \int_\Gamma (x, \gamma)F(\gamma)d\gamma \qquad (x \in G, F \in L^1(\Gamma)),$$

where Γ is the dual group of G and (x, γ) is the value of the character γ at the point x. With the norm

$$\|f\| = \int_\Gamma | F(\gamma) | d\gamma$$

$A(G)$ is a commutative Banach algebra, and G is its maximal ideal space.

If I is a closed ideal in $A(G)$, let $Z(I)$ be the set of all $x \in G$ such that $f(x) = 0$ for every $f \in I$. Malliavin [**3**; **4**; **5**] has recently solved a problem of long standing by proving that in every nondiscrete G there is a closed set E such that $E = Z(I_1) = Z(I_2)$ for two *distinct* closed ideals I_1 and I_2 in $A(G)$. Combined with an older result of Helson [**1**] this implies that there are infinitely many closed ideals I in $A(G)$ with $Z(I) = E$.

It is the purpose of this note to point out that Malliavin's construction for compact G (he reduced the general case to this) yields an even more specific result:

THEOREM. *Suppose G is an infinite compact abelian group. There is a real $f \in A(G)$ such that the closed ideals I_n generated by the powers f^n $(n = 1, 2, 3, \cdots)$ are all distinct.*

We sketch the proof. If $g \in A(G)$ and u is a real number, we define $a_\gamma(u)$ by

$$e^{iug(x)} = \sum_{\gamma \in \Gamma} a_\gamma(u) \cdot (x, \gamma) \qquad (x \in G). \tag{1}$$

Malliavin [**5**] constructed a real $g \in A(G)$ for which

$$| a_\gamma(u) | < \exp(-C | u |^{1/2}) \qquad (\gamma \in \Gamma), \tag{2}$$

where $C > 0$ is independent of γ. (The exponent 1/2 in (2) could be

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replaced by any $\lambda<1$, but not by 1. Kahane's construction [2] should also be mentioned in this connection.) By (2),

$$\sup_{\gamma\in\Gamma}\int_{-\infty}^{\infty}|a_\gamma(u)u^n|\,du = M_n < \infty \qquad (n = 0, 1, 2, \cdots). \tag{3}$$

The mapping

$$\phi \to \int_G \phi(g(x))(-x, \gamma)dx \tag{4}$$

is, for each γ, a bounded linear functional in the space of all continuous functions ϕ on the range of g, and hence there are measures μ_γ on the line, with compact support, such that

$$\int_G \phi(g(x))(-x, \gamma)dx = \int_{-\infty}^{\infty}\phi(t)d\mu_\gamma(t). \tag{5}$$

Taking $\phi(t)=e^{iut}$, we see that $a_\gamma(u)$ is the Fourier-Stieltjes transform of μ_γ, and (3) implies that $d\mu_\gamma(t)=m_\gamma(t)dt$, where each m_γ is infinitely differentiable and

$$\left| m_\gamma^{(n)}(t)\right| \leqq M_n \qquad (\gamma\in\Gamma,\ t \text{ real}). \tag{6}$$

Since $a_0(0)=1$, $m_0\not\equiv 0$, and there is a real number α such that $m_0(\alpha)\neq 0$.

Put $f(x)=g(x)-\alpha$. By (6), the expressions

$$T_n h = (-1)^n \sum_{\gamma\in\Gamma} H(\gamma)m_\gamma^{(n)}(\alpha) \qquad (n = 1, 2, 3, \cdots), \tag{7}$$

where $h(x)=\sum H(\gamma)(x, \gamma)$, define bounded linear functionals on $A(G)$. The following two facts show that T_n annihilates I_{n+1} but not I_n, and hence establish the theorem:

(A) $T_n f^n \neq 0$.

(B) If $h(x)=(x, \gamma_0)f^{n+1}(x)$, for any $\gamma_0\in\Gamma$, then $T_n h=0$.

(A) and (B) are proved by evaluating (7) for all h of the form

$$h(x) = P(g(x))(x, -\gamma_0) \qquad (\gamma_0\in\Gamma) \tag{8}$$

where P is a polynomial. Set

$$c_{j,n}(\gamma) = \int_{-\infty}^{\infty} W_j^{(n)}(t)m_\gamma(t)dt, \tag{9}$$

where $\{W_j\}$ is a sequence of non-negative infinitely differentiable functions which vanish outside $(\alpha-1/j, \alpha+1/j)$, such that $\int_{-\infty}^{\infty} W_j(t)dt$

$=1$. Integrating (9) by parts n times, we see that $|c_{j,n}(\gamma)| \leqq M_n$ and $\lim_j c_{j,n}(\gamma) = (-1)^n m_\gamma^{(n)}(\alpha)$. Hence (5) implies, if h is of the form (8), that

$$\begin{aligned} T_n h &= \lim_j \sum_\gamma H(\gamma) \int_{-\infty}^{\infty} W_j^{(n)}(t) m_\gamma(t)\,dt \\ &= \lim_j \sum_\gamma H(\gamma) \int_G W_j^{(n)}(g(x))(x, \gamma)\,dx \\ &= \lim_j \int_G W_j^{(n)}(g(x)) P(g(x))(x, -\gamma_0)\,dx \\ &= \lim_j \int_{-\infty}^{\infty} W_j^{(n)}(t) P(t) m_{\gamma_0}(t)\,dt \\ &= (-1)^n \lim_j \int_{-\infty}^{\infty} W_j(t) \left(\frac{d}{dt}\right)^n [P(t) m_{\gamma_0}(t)]\,dt \\ &= (-1)^n \left(\frac{d}{dt}\right)^n [P(t) m_{\gamma_0}(t)]_{t=\alpha}. \end{aligned}$$

Taking $h=(g-\alpha)^n$, it follows that $T_n f^n$ is the nth derivative of $(-1)^n(t-\alpha)^n m_0(t)$, evaluated at $t=\alpha$, and this is $(-1)^n n! m_0(\alpha) \neq 0$. This proves (A).

Taking $h(x)=(x, \gamma_0)(g(x)-\alpha)^{n+1}$, we see that $T_n h$ is the nth derivative of $(-1)^n(t-\alpha)^{n+1} m_{\gamma_0}(t)$, evaluated at $t=\alpha$, which is 0. This proves (B).

References

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