THE FEBRUARY MEETING IN NEW YORK

The four hundred eighty-ninth meeting of the American Mathematical Society was held at Hunter College on Saturday, February 28. There were almost 300 persons in attendance, including the following 246 members of the Society:

C. R. Adams, M. I. Aissen, R. G. Albert, A. G. Anderson, R. L. Anderson, R. G. Archibald, P. N. Armstrong, N. J. Artin, Silvio Aurora, W. G. Bade, Joshua Barlaz, J. H. Barrett, L. K. Barrett, R. G. Bartle, J. D. Baum, Anatole Beck, E. G. Begle, Lipman Bers, Nicholas Bilotta, D. W. Blackett, E. K. Blum, Samuel Borofsky, S. G. Bourne, J. W. Bower, C. B. Boyer, J. W. Brace, A. D. Bradley, Roslyn Braverman, F. E. Browder, A. B. Brown, C. T. Bumer, J. H. Bushey, Jewell H. Bushey, F. P. Callahan, P. W. Carruth, Joshua Chover, W. L. Chow, L. W. Cohen, Richard Courant, L. M. Court, V. F. Cowling, M. L. Curtis, P. C. Curtis, Jr., J. H. Curtiss, M. D. Darkow, D. A. Darling, R. B. Davis, J. C. E. Dekker, V. J. Doberly, Jesse Douglas, James Dugundji, Nelson Dunford, Jaques Dutka, J. E. Eaton, Samuel Eilenberg, C. C. Elgot, Bernard Epstein, M. P. Epstein, Paul Erdös, M. E. Estill, Trevor Evans, W. H. Fagerstrom, J. M. Feld, Chester Feldman, F. G. Fender, W. E. Ferguson, Irwin Fischer, R. M. Foster, Gerald Freilich, Orrin Frink, M. P. Gaffney, G. N. Garrison, Murray Gerstenhaber, H. A. Giddings, B. P. Gill, Wallace Givens, Sidney Glusman, Samuel Goldberg, S. I. Goldberg, Daniel Gorenstein, Morikuni Goto, H. S. Grant, L. W. Green, F. L. Griffin, H. M. Griffin, Emil Grosswald, Laura Guggenbuhl, Felix Haas, Sister M. Raphael Hafner, Carl Hammer, Alvin Hausner, G. A. Hedlund, Alex Heller, R. T. Herbst, Aaron Herschfeld, David Hertzig, L. S. Hill, I. I. Hirschman, F. E. P. Hirzebruch, S. P. Hoffman, Banesh Hoffmann, Alfred Horn, Alfred Huber, Ralph Hull, T. R. Humphreys, D. H. Hyers, B. M. Ingersoll, H. G. Jacob, R. C. James, Shizuo Kakutani, Aida Kalish, Yukiyosi Kawada, M. E. Kellar, J. F. Kiefer, H. S. Kieval, M. S. Klamkin, George Klein, I. I. Kolodner, B. O. Koopman, Saul Kravetz, J. B. Kruskal, Jr., M. D. Kruskal, R. R. Kuebler, A. W. Landers, M. K. Landers, P. D. Lax, Solomon Leader, Solomon Lefschetz, Benjamin Lepson, J. J. Levin, D. J. Lewis, S. D. Liao, H. M. Lieberstein, M. A. Lipschutz, E. R. Lorch, D. B. Lowdenslager, Eugene Lukacs, R. C. Lyndon, B. H. McCandless, L. A. MacColl, H. M. MacNeille, Wilhelm Magnus, Irwin Mann, A. J. Maria, M. H. Maria, Imanuel Marx, A. P. Mattuck, A. E. Meder, Jr., L. F. Meyers, K. S. Miller, W. H. Mills, Don Mittleman, J. C. Moore, Morris Morduchow. G. D. Mostow, Simon Mowshowitz, E. R. Mullins, Jr., D. S. Nathan, C. A. Nelson, L. W. Neustadt, Morris Newman, Katsumi Nomizu, A. B. Novikoff, C. O. Oakley, R. E. O'Donnell, A. F. O'Neill, J. C. Oxtoby, J. S. Oxtoby, T. A. Paley, F. P. Pedersen, Everett Pitcher, H. O. Pollak, Walter Prenowitz, M. H. Protter, Howard Raiffa, D. B. Ray, G. E. Raynor, Helene Reschovsky, H. G. Rice, Moses Richardson, C. E. Rickart, E. K. Ritter, I. F. Ritter, Robin Robinson, Selby Robinson, David Rosen, M. A. Rosenlicht, J. E. Rosenthal, J. P. Russell, Charles Salkind, Hans Samelson, J. E. Sammet, Jacob Samoloff, J. E. Sanders, Arthur Sard, Shigeo Sasaki, R. D. Schafer, J. A. Schatz, Samuel Schecter, Pincus Schub, Abraham Schwartz, C. H. W. Sedgewick, R. R. Seeber, Jr., R. J. Semple, H. N. Shapiro, H. S. Shapiro, V. L. Shapiro, James Singer, P. A. Smith, J. J. Sopka, W. K. Spears, E. P. Starke, J. S. Stubbe, M. M. Sullivan, G. L. Thompson, D. L. Thomsen, Jr., M. L. Tomber, P. M. Treuenfels, A. W. Tucker, H. G. Tucker, R. J. Turyn, D. H. Wagner, Sylvan

Wallach, M. T. Wechsler, J. V. Wehausen, J. H. Weiner, Louis Weisner, John Wermer, M. E. White, A. L. Whiteman, P. M. Whitman, Albert Wilansky, J. E. Wilkins, Jr., A. B. Willcox, K. G. Wolfson, V. M. Wolontis, M. A. Wurster, Fumio Yagi, Hidehiko Yamabe, Michael Yanowitch, Bertram Yood, D. M. Young, E. H. Zarantonello, J. A. Zilber, H. J. Zimmerberg, Leo Zippin.

An address, *On the foundations of algebroid geometry*, was presented at the General Session by Professor W. L. Chow of The Johns Hopkins University by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. Professor C. R. Adams presided.

Sessions for contributed papers were held in the morning and in the afternoon. Professors R. M. Foster and Louis Weisner presided in the morning and Professors Paul Erdös and Everett Pitcher in the afternoon.

Coffee was served by students and members of the Department of Mathematics after the General Session.

Abstracts of the papers presented follow, those with "*t*" after the abstract number having been presented by title. Of the papers having joint authorship paper 286 was presented by Miss Scheerer, paper 295 by Dr. Lukacs, and paper 304 by Professor Young. Dr. Kalaba was introduced by Dr. Robert Davies and Mr. Darsow by Professor Irving Kaplansky.

ALGEBRA AND THEORY OF NUMBERS

266*t*. Volodymyr Bohun-Chudyniv: *On a method of determining orthogonal square matrices of* $(2^{1+\lambda}-1)2^K(\lambda \geqq 1,\ K \geqq 1;\ \lambda=0,\ K>2)$ *order, composed of differing integers.*

In the works of: (1) J. J. Sylvester, *Thoughts on inverse orthogonal matrices* etc., Philosophical Magazine (4) vol. 34 (1867) pp. 461–475; (2) A. C. Paley, *On orthogonal matrices*, Journal of Mathematics and Physics vol. 12 (1933) pp. 311–320, are given methods of determining orthogonal square matrices of 2^K $(K \geqq 2)$ order and in Paley's work also ones of other orders (see table I, p. 317 of his work), but elements of these matrices can only be ± 1. In author's paper *On orthogonal and non-orthogonal closed systems of K-nions and their application* (Bull. Amer. Math. Soc. Abstract 58-6-562) is given a method of determining closed systems of K-nions and a method of constructing with their help orthogonal matrices of $(2^{1+\lambda}-1)$ $(\lambda>1)$ and 2^K $(K>2)$ orders composed of differing integers. In the present paper the author determines the method of constructing orthogonal square matrices of $(2^{1+\lambda}-1)2^K$ $(\lambda \geqq 1,\ K \geqq 1;\ \lambda=0,\ K>2)$ order, elements of which are also differing integers. Matrices of this type we determine with the help of 2 lemmas and schemes of K-nions, constructed by the author for determining orthogonal square matrices of $(2^{1+\lambda}-1)$ and 2^K orders. A particular case of the first lemma is the first lemma of Paley. The largest number of elements in these matrices can equal $3\cdot(2^{1+\lambda}-1)^2$ for the first group of matrices and $3\cdot 2^{2K}$ for the second. The Sylvester-Paley orthogonal square matrices are included in author's matrices as a particular case. (Received January 14, 1953.)

267t. Leonard Carlitz: *A note on partitions in $GF[x, q]$.*

Problems of the following kind are discussed. For $A \in GF[x, q]$, $A \neq 0$, let $p(A)$ denote the number of solutions of $A = U_1 + U_2 + \cdots$, $\deg A = \deg U_1 > \deg U_2 > \cdots$; let $p_k(A)$ denote the number of solutions in which there are precisely k summands. Then it is shown that if $\deg A = m$, then $p(A) = \prod_{i=0}^{m-1} (1 + (q-1)q^i)$, while $p_k(A) = q^{(k-1)(k-2)/2}\left[{m \atop k-1}\right]$. (Received January 12, 1953.)

268t. Leonard Carlitz: *Note on some partition identities.*

The paper contains elementary proofs of some identities proved by Newman, *Remarks on some modular identities*, Trans. Amer. Math. Soc. vol. 37 (1952) pp. 313–320, as well as of some similar formulas. (Received January 12, 1953.)

269t. Leonard Carlitz: *The first factor of the class number of a cyclic field.*

Let $p-1 = ab$, where b is odd and >1. Let $\zeta = e^{2\pi i/p}$, and let $K \subset R(\zeta)$ denote the cyclic field of degree a over the rational field; let h_a denote the first factor of the class number of K. It is shown that $h_a \equiv 2^{-(a-2)/2} \prod B_{bup^n+1} \pmod{p^n}$, where $u = 1, 3, \cdots, a-1$. The method is that used by Vandiver, *On the first factor of the class number of a cyclotomic field*, Bull. Amer. Math. Soc. vol. 25 (1918–19) pp. 458–461. (Received January 12, 1953.)

270t. Eckford Cohen: *The finite Goldbach problem in algebraic number fields.*

In a previous paper (Bull. Amer. Math. Soc. Abstract 58-4-327) the author gave a criterion for the representability of the elements of the ring $R(m)$ of residue classes modulo an integer $m > 1$, as sums of prime elements of $R(m)$. This result is extended to an arbitrary finite extension F of the rationals. Let $A \neq 1$ be an integral nonzero ideal of F and suppose that A has m distinct prime ideal factors, of which h are of norm >2 and k are of norm $=2$, $(h+k=m)$. If A has but a single prime divisor Q of norm 2 $(k=1)$, denote by μ the maximum power to which Q divides A. The following theorem is proved: *If $R(A)$ denotes the ring of residue classes* (mod A), *then there exists an n such that all elements of $R(A)$ are expressible as a sum of n primes of $R(A)$ if and only if $m > 1$, $h > 0$. For such A, the minimum value M of n is given by $M = 2$ if $k = 0$ and $h \geqq 2$; by $M = 3$ if $k = 1$ and $h \geqq 2$, if $k = 1$, $h = 1$, and $\mu = 1$, or if $k = 2$ and $h \geqq 1$; by $M = 4$ if $k = 1$, $h = 1$, and $\mu > 1$; and by $M = k$ if $k \geqq 3$ and $h \geqq 1$.* Other results of a related nature are also proved. (Received February 12, 1953.)

271t. Eckford Cohen: *The number of solutions of quartic congruences in two unknowns.*

Using an approach employed previously in the case of cubic congruences, the author obtains explicit formulas for the number of solutions of the quartic congruences (I) $ax^4 + by^4 \equiv n \pmod{p^\lambda}$ and (II) $ax^4 + by^2 \equiv n \pmod{p^\lambda}$, where p is a prime $\equiv 1 \pmod 4$, $(a, p) = (b, p) = 1$, and n is arbitrary. The formulas obtained for (I) and (II) are particularly simple in case $t \not\equiv 0 \pmod{p^\lambda}$, t being defined by $n = p^t \xi$, $(\xi, p) = 1$. On the basis of these results, solvability criteria for (I) and (II) are derived. For example, it is shown, *in case $p \equiv 1$* (mod 8), *$p \neq 17$, that* (I) *is insolvable if and only if $t \not\equiv 0$* (mod 4), *$t < \lambda$, and a and b have a different quartic character* (mod p), *and in case $p \equiv 5$* (mod 8), *$p > 29$, that* (I) *is insolvable if and only if $t \not\equiv 0$* (mod 4), *$t < \lambda$, and a and b either have the same quartic character or have a different quadratic character* (mod p). Formulas for the number of primitive solutions of (I) and (II) in the case of a prime modulus ($\lambda=1$) are also found. (Received February 12, 1953.)

272. V. J. Doberly: *Doberly's series and products.*

Through disintegrating the product of any number (n) of successive terms of arithmetical progression a, $a+h$, $a+2h, \cdots$, $[a+(n-1)h]$ into an algebraical series, all terms of which are proved to be expressible as $(n-2)!$ $\cdot(a+h)\sum_{m=1}^{m=n-1} a(a+h)(a+2h)\cdots[a+(m-1)h]h^{n-m-1}/(n-1)!$, the author has found many new types of final series, all easily yielding to summation through a simple algebraical expression consisting of one single term only. Typical among the simplest of the series permitting such summation of any number of their terms are numerical successions similar to $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\cdots$ and $3+3\cdot 4/1!+3\cdot 4\cdot 5/2!$ $+3\cdot 4\cdot 5\cdot 6/3!+\cdots$ as well as the progression of a given definite number of terms with successively decreasing powers in one of their factors as in the series $3\cdot 2^3$ $+3\cdot 5\cdot 2^2/1!+3\cdot 5\cdot 7\cdot 2^1/2!+3\cdot 5\cdot 7\cdot 9\cdot 2^0/3!$, where $n=4$. (Received January 9, 1953.)

273. Alfred Horn: *Doubly stochastic matrices and the diagonal of an orthogonal matrix.*

If $x_1\geqq\cdots\geqq x_n$, $y_1\geqq\cdots\geqq y_n$, and if $\sum_{i=1}^k x_i\leqq\sum_{i=1}^k y_i$, $1\leqq k\leqq n-1$, and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, then there exists a real orthogonal matrix (a_{ij}) such that x_i $=\sum_{j=1}^n a_{ij}^2 y_j$. This generalizes a well known theorem of Hardy, Littlewood, and Pólya. As an application it is shown that a vector x is the diagonal of a real orthogonal matrix with determinant $+1$ if and only if it lies in the convex hull of those points $(\pm 1,\cdots,\pm 1)$ of which an even number of coordinates are -1. It follows that a vector x can be the diagonal of a real orthogonal matrix if and only if $\sum_{j\neq i}|x_j|$ $\leqq n-2+|x_i|$ and $|x_i|\leqq 1$, $1\leqq i\leqq n$. If $x_i\geqq 0$, these last conditions are also necessary and sufficient for the diagonal of a doubly stochastic matrix. (Received January 15, 1953.)

274*t*. J. R. Isbell: *Birkhoff's problem* 111.

A doubly stochastic infinite matrix is a limit of linear combinations of permutation matrices, error being measured by the least upper bound of sums in rows and columns of absolute errors, if and only if for every $\epsilon>0$ there is n such that in every row and column the sum of the largest n elements is $>1-\epsilon$. In other norms there are unsatisfactory trivialities. The fundamental tool is König's selection theorem. This is a substantially complete solution of the last problem in Birkhoff's well known list. (Received January 15, 1953.)

275*t*. J. R. Isbell: *On finitary games.*

The fundamental results of the theory of non-cooperative games are extended to two classes of gamelike processes. A finite Kuhn diagram a play of which may intersect one information set up to n times is fully equivalent, if players are permitted to randomize at each move, to a continuous game on a product of cells with payoff a polynomial of the nth degree; all polynomials in appropriate variables are such payoffs. A stochastic and competitive process with payoff only at absorbing states and all transition probability distributions atomic has stationary equilibrium points; if it is two-person zero-sum, the payoff in the stationary strategies is linear fractional and the sets of good strategies are convex. A game of the latter type has an essential subgame with a closed solution satisfying the dimensionality relation, and its solution is equivalent to the solution of a system of matrix games; but inessential pure strategies may figure in good strategies. (Received January 15, 1953.)

276. H. G. Jacob, Jr.: *A theorem on Kronecker products.*

By defining the notion of rank and coherence for elements of a Kronecker product of vector spaces of arbitrary dimensions the results of L. K. Hua (Sci. Rep. Nat. Tsing Hua Univ. Ser. A vol. 5 (1948) pp. 150–181) may be generalized. Let $\mathfrak{X}_i$, $\mathfrak{Y}_i$, $i=1, 2$, be right and left vector spaces respectively over the division rings Δ_i. Let $\mathfrak{X}_i \times_i \mathfrak{Y}_i$ denote the Kronecker products of these spaces and $\mathfrak{Y}_i \otimes_i \mathfrak{X}_i$ denote the products of the same spaces regarded as right and left vector spaces respectively over Γ_i, the anti-isomorphic images of Δ_i. Let $T=\sum_{j=1}^n x_j \times_1 y_j \in \mathfrak{X}_1 \times_1 \mathfrak{Y}_1$ and $S=\sum_{j=1}^n y_j \otimes_1 x_j \in \mathfrak{Y}_1 \otimes_1 \mathfrak{X}_1$. Theorem: A necessary and sufficient condition that the mapping $T \rightarrow T^\sigma$ of $\mathfrak{X}_1 \times_1 \mathfrak{Y}_1$ onto $\mathfrak{X}_2 \times_2 \mathfrak{Y}_2$ be one-to-one, coherence invariant, and possesses a coherence invariant inverse is that either (1) $T^\sigma=\sum_{j=1}^n x_j P \times_2 y_j Q + R$, where P and Q are semi-linear transformations of $\mathfrak{X}_1$ onto $\mathfrak{X}_2$ and $\mathfrak{Y}_1$ onto $\mathfrak{Y}_2$ respectively, with the same isomorphisms of Δ_1 onto Δ_2, and $R \in \mathfrak{X}_2 \times_2 \mathfrak{Y}_2$; or (2) $T^\sigma = S^\gamma$, where $T \rightarrow S$ is the natural mapping of $\mathfrak{X}_1 \times_1 \mathfrak{Y}_1$ onto $\mathfrak{Y}_1 \otimes_1 \mathfrak{X}_1$ and $S^\gamma = \sum_{j=1}^n y_j Q \times_2 x_j P + R$ where Q and P are semilinear transformations of $\mathfrak{Y}_1$ onto $\mathfrak{X}_2$ and $\mathfrak{X}_1$ onto $\mathfrak{Y}_2$ respectively, with same isomorphism, and R is as described above. (Received January 15, 1953.)

277. D. J. Lewis: *Singular quartic forms.*

In the study of forms over a $\mathfrak{p}$-adic field K, it is well known that if a form has a nonsingular zero modulo $\mathfrak{p}$ in K, it has a nontrivial zero in K. Thus it is a natural problem to study forms which have only singular zeros modulo $\mathfrak{p}$ in K; or equivalently, forms over a finite field k which have only singular zeros in k. The author has previously (Ann. of Math. vol. 56 (1952) pp. 473–478) exhibited and made use of properties of such forms of degree three. In this paper it is shown that if a quartic form over $k=GF(q)$ has only singular zeros in k, then if $2\nmid q$ and q is sufficiently large, the form must be of one of the following types: (i) $e(Q_1^2-vQ_2^2)$, (ii) $e(L_1^2-vL_2^2)(L_3^2-uL_4^2)$, or (iii) $G(L_1, L_2, \cdots, L_s)$, where $e=\pm 1$, v and u are either zero or non-squares of k, Q_i and L_i are respectively quadratic and linear forms over k, and $G(z_1, z_2, \cdots, z_s)$ is a quartic form having only the trivial zero in k, hence $s \leqq 4$. An essential step in the proof utilizes a generalization of a theorem of Carlitz (Duke Math. J. vol. 19 (1952) pp. 471–474). (Received January 12, 1953.)

278t. R. C. Lyndon: *Identities in finite algebras.*

A nonassociative multiplication is defined over a finite set S, with the property that there exists no finite set of identities for S from which all others are derivable. (Received January 19, 1953.)

279. A. B. Willcox: *Some structure theorems for a class of Banach algebras.*

Let R be a B-algebra with unit and whose Stone-Jacobson structure space, $S(R)$, is Hausdorff. If R is commutative this is equivalent to R being regular in the sense of Silov [Travaux de l'Inst. Math. Stekloff, 1947]. For B-algebras with H.s.s. this paper presents some theorems on representation of closed ideals as intersections of primitive ideals or of closed primary ideals. Some of these are generalizations of theorems due to Silov in the regular-commutative case. Also generalized from Silov's paper are the basic theorems on separation of closed disjoint sets in $S(R)$ and on existence of minimal closed primary ideals $\overline{J(P)}$, P any primitive ideal. Using the ideals $\overline{J(P)}$ the definition of "type C" is the obvious generalization of Silov's definition, and R is of type C_0 if it is of type C and if the functions $\|x(P)\|$ are continuous on $S(R)$ for all $x \in R$. If R, of type C, has H.s.s. then R is a sub-direct sum of the primary B-algebras $R/\overline{J(P)}$, and the sum has certain continuity properties if R is of type C_0. The reverse process of constructing B-algebras of type C or C_0 from arbitrary primary B-algebras is possible under some continuity restrictions on the sums used, and this process is, in a sense, unique if the elements of the sum distinguish between the elements of the Hausdorff space of indices. (Received January 9, 1953.)

280. K. G. Wolfson: *An ideal-theoretic characterization of the ring of all linear transformations.*

Let K be an abstract ring, and call a left ideal of K a left annulet if it is the totality of left annihilators of a subset of K. Then K is isomorphic to a ring $T(F, A)$ of all linear transformations of the vector space A over the division ring F if and only if (1) K_0, the socle of K, is not a zero-ring, and is contained in every nonzero two-sided ideal of K. (2) Any left ideal of K which is annihilated on the right only by zero contains K_0. (3) The sum of two left annulets is a left annulet. (4) K contains an identity element. The method of proof depends on the existence of a Galois-type correspondence between the lattice of annulets of a ring of linear transformations, and the lattice of subspaces of the underlying vector space. A characterization of all two-sided ideals of $T(F, A)$ is also given. (Received January 13, 1953.)

Analysis

281*t*. Joseph Andrushkiw: *A note on power series whose partial sums have only real zeros.*

Let (a_n), $n=0, 1, 2, \cdots, a_0=1$, be a sequence of real numbers such that $f_n(z) = 1+\sum_{k=1}^{n} a_k z_k$, $n>N$, is a polynomial whose zeros are either all positive or all negative. Since the derivative $f_n^{(n-2)}(z)$ has also only real zeros, it follows that $|a_n|^{1/n} < 2^{(1-n)/2}a_1$, $n>N$, and hence that the power series $f(z)=1+\sum_{k=1}^{\infty} a_k z_k$ is convergent for all values of z. The polynomial $f_n(z)$, $n>N$, possessing at most double zeros (Andrushkiw, Bull. Amer. Math. Soc. Abstract 58-2-151), one can apply the theorem of Grommer (J. Reine Angew. Math. vol. 144 (1914) pp. 114–165) in modified form (Andrushkiw, Bull. Amer. Math. Soc. Abstract 58-3-289) and conclude that $f(z)$ is an integral function with only positive (negative) zeros and of genus 0, except for the factor e^{cz}, $c \leqq 0$ $(c \geqq 0)$. If $f_n(z)$, $n>N$, has only real (but not necessarily of the same sign) zeros, it can be shown that $|a_n|^{1/n} < n^{-1/2}(a_1^2-2a_2)^{1/2}$, $n>N$. In a similar way it follows that $f(z)$ is an integral function with only real zeros and of genus 1, except for the factor e^{cz^2} $(c \leqq 0)$. (Received January 6, 1953.)

282. E. K. Blum: *The fundamental group of the principal component of a commutative Banach algebra.*

Let B denote an arbitrary commutative Banach algebra over the complex numbers. It is assumed that B contains a unit element, e, with norm equal to 1. If b is an element in B such that b^{-1} exists $(bb^{-1}=b^{-1}b=e)$, then b is "regular." Let G denote the set of regular elements. G is an open set, hence is a union of maximal open connected sets, its components. Let G_1 be the component which contains e. Now, the function $\exp(x) \equiv e+\sum_{n=1}^{\infty} x^n/n!$ is defined for all x in B and has the usual properties of the classical exponential function. If $\prod_1(G_1)$ denotes the fundamental group of G_1, the main result of this paper may be stated as follows: *Theorem.* Let P be the set of all x in B such that $\exp(x)=e$. Then P is an additive group which is isomorphic to

$\prod_1(G_1)$. Two proofs are given. The first is based on Schreier's theory of the universal covering group. The second depends only on results from the theory of Banach algebras. (Received January 12, 1953.)

283. F. E. Browder: *The existence of a fundamental solution in the large for the general linear elliptic system of differential equations with analytic coefficients.*

Let D be a bounded domain in E^n, r and m positive integers, $C_c^{2m}(D)$ the family of $2m$-times continuously differentiable r-vector functions with compact support in D. The system of r differential operators of order $2m$, $K_i = \sum_{k_1 \cdots k_{2m}, j} a_{k_1 \cdots k_{2m}; i, j}(x)\partial^{2m}/\partial x_{k_1} \cdots \partial x_{k_{2m}} + R_i$, is said to be linear elliptic system on D if each of the R_i is a linear differential operator on D of order less than $2m$ acting on $u_1, \cdots u_r$ and if in addition $\sum_{k_1 \cdots k_{2m}} a_{k_1 \cdots k_{2m}; i, j}(x)\xi_{k_1} \cdots \xi_{k_{2m}}$ is a nonsingular $(r \times r)$-matrix for $x \in D$ and any real n-vector (ξ). By a fundamental solution matrix for such a system is meant an $(r \times r)$-matrix $e_{ij}(x, z)$ whose entries are defined for $x, z \in D$ and $x \neq z$ and such that $u_i(x) = \sum_j \int_D e_{ij}(z)\overline{K}_j(u)dz$ for $(u) \in C_c^{2m}(D)$ ($\overline{K}$ the adjoint system to K). It is shown that if the coefficients of K are analytic on D, then there always exists a fundamental solution for K on D, each of whose entries is analytic in x and in z for $x, z \in D$ and $x \neq z$. The proof uses a device of Holmgren, usually employed in showing the uniqueness of the solution of the Cauchy problem. (Received January 14, 1953.)

284*t*. P. L. Butzer and Waclaw Kozakiewicz: *A theorem on the generalized derivatives.*

Let the function $f(x)$ be defined in the open interval (a, b), then the "right" difference of order $s+1$ is defined inductively by the relations $\Delta_h^1 f(x) = f(x+h) - f(x)$, $\Delta_h^{s+1} f(x) = \Delta_h^s f(x+h) - \Delta_h^s f(x)$, $h > 0$. Among the results obtained is the following: If $f(x)$ is continuous in (a, b) and if there exists a sequence $\{h_n\}$ of positive numbers converging to zero and a function $s(x)$ Lebesgue integrable in every closed subinterval of (a, b) such that $\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} [\Delta_{h_n}^{l+1} f(x)]/h_n^{l+1} = 0$ almost everywhere in (a, b) and $|s_n(x)| \leqq s(x)$, $n = 1, 2, 3, \cdots$, for $a < x < x + (l+1)h_n < b$, then $f(x)$ is a polynomial of degree l in (a, b). This result contains as a particular case one due to Th. Anghelutza (Bull. Sci. Math. (2) vol. 63 (1939) pp. 239–246). A similar theorem can be stated for the "central" difference of order $s+1$ defined inductively by $\nabla_{2h}^1 f(x) = f(x+h) - f(x-h)$, $\nabla_{2h}^{s+1} f(x) = \nabla_{2h}^s f(x+h) - \nabla_{2h}^s f(x-h)$, $h > 0$. The method of proof consists of an extensive use of integral operators. (Received January 6, 1953.)

285*t*. W. F. Darsow: *On positive definite functions and pure states.*

For a locally compact group G with its Haar measure, let Q be the set of positive definite functions on G that can be uniformly approximated on compact sets by functions of the form $f * \tilde{f}$ where f is a continuous function on G vanishing outside some compact set and $\|f\|_2 = 1$; and let A be the uniform closure of the set of all bounded linear operators L_f on L^2 with f in L^1 where $L_f g = f * g$. It is shown that there exists a convex-linear isomorphism between the set of states of A and Q. Furthermore, a discrete, countable group is exhibited for which the constant function 1 does not belong to Q. This answers in the negative problem 5 of R. Godement (Trans. Amer. Math. Soc. vol. 63 (1948) pp. 1–84). As a consequence it follows that the irreducible, unitary representations arising from pure states in the manner of I. E. Segal (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 73–88) do not in general exhaust all the (strongly continuous) irreducible, unitary representations. (Received January 6, 1953.)

286. Bernard Epstein and Anne Scheerer: *Brownian motion and the Green's function. The plane case.*

M. Kac has recently established (Proceedings of the Second Berkeley Symposium) the existence of the exterior Green's function of a domain Ω in three-dimensional space by means of the theory of Brownian motion (Wiener measure). While the method is easily extended to higher dimensions, serious difficulties arise in the two-dimensional case, due to the divergence of certain integrals. By introducing a parameter ϵ which renders the integrals convergent, and carrying out certain limiting operations, Kac's procedure can be modified so as to establish the existence of the exterior Green's function (with pole at infinity) of a bounded plane domain whose boundary satisfies certain mild conditions. The chief difficulty that has to be overcome is that the probabilistic interpretation of certain functions has to be abandoned, thus necessitating a proof by analytic methods of certain inequalities which are obvious by probability arguments in the three-dimensional case. (Received January 15, 1953.)

287t. Herbert Federer: *An integralgeometric theorem.*

Let H^k be the k-dimensional Hausdorff measure over Euclidean n-space and let μ be a Haar measure over the group of isometries of n-space. Suppose A and B are analytic subsets of n-space, A is (H^k, k)-rectifiable, B is countably l-rectifiable [for definitions of these terms see H. Federer, *The* (ϕ, k)*-rectifiable subsets of n-space,* Trans. Amer. Math. Soc. vol. 62 (1947) p. 126], and suppose $m=k+l-n\geqq 0$. Then $A\cap f(B)$ is (H^m, m)-rectifiable for μ almost all isometries f, and $\int H^m[A\cap f(B)]d\mu f=c\cdot H^k(A)\cdot H^l(B)$ where c depends only on k, l, n, and μ. [The proofs given in the paper use a particular Haar measure μ and yield the corresponding value of c explicitly in terms of k, l, and n.] (Received January 15, 1953.)

288. Murray Gerstenhaber: *Singularities on the varieties of modules of Riemann surfaces.*

The conjectures which Teichmueller made public in 1939 concerning the problem of the modules enable one to conclude that if an algebraic variety (over the complex numbers) can be constructed, the points of which, after the deletion of a subvariety, are in one-to-one correspondence with the conformal equivalence classes of Riemann surfaces of genus g, then this variety must have singular points whenever g is greater than one. In particular, that point of the variety which corresponds to the unique hyperelliptic Riemann surface of genus g which possesses a conformal self-transformation of order $2g+1$ will be singular; the variety will in fact not be locally euclidean at that point. This singularity will be isolated if and only if $2g+1$ is prime. Until Teichmueller's conjectures are proved, these conclusions are also only conjectures, but in the case of genus two they have been verified independently by algebraic methods by Max Rosenlicht and by transcendental methods by the author. It is further indicated by the Teichmueller conjectures that, in the case of genus 2, the singular point described in the foregoing is unique. (Received January 12, 1953.)

289t. Seymour Ginsburg: *Incomparable order types.*

Pairs of simply ordered sets, whose symmetric difference contains just two elements and whose order types are incomparable, are studied. The principal result obtained is the following. Let p and q, $p<q$, be two fixed points in the set $A\cup\{p, q\}$. If the set $\{x \mid p<x<q, x\in A\}$ is infinite, then the order types of $A\cup\{p\}$ and $A\cup\{q\}$ are incomparable. The converse is also true, namely, if B and C have incomparable

order types, and if the symmetric difference of B and C contains just two elements, p and q, where $p<q$ in $D=B\cup C$, then (1) p and q are both fixed points in D; (2) $B=A\cup\{y\}$ and $C=A\cup\{z\}$, where y is one of the points p or q, and z is the other point; and (3) the set $\{x|p<x<q, x\in D\}$ is infinite. (Received January 12, 1953.)

290. Felix Haas: *The Poincaré-Bendixon theorem for closed two-dimensional manifolds different from the torus.*

In the following M is a two-dimensional, closed, orientable, manifold. V is a vector field on M which satisfies a Lipschitz condition and has at most a denumerable number of singular points. C^+ is a characteristic of V. $\overline{C}$, the set of ω-limit points of C^+, is assumed to be free of singular points. Then the following theorem holds: Unless M is a torus and V is free of singular points, $\overline{C}$ is a closed curve; C^+ either equals $\overline{C}$ or spirals toward $\overline{C}$ from one side. In the proof it is first shown that a closed curve S belongs to $\overline{C}$. It is then shown that if $\overline{C}$ is nowhere dense on M, then S is a characteristic. By a previous result of the author this implies that S is a characteristic unless M is a torus and V is free of singular points. Next it is proved that under the same hypothesis $\overline{C}$ contains no points which do not belong to S. And finally, it is shown that C^+ either equals $\overline{C}$ or spirals toward $\overline{C}$ from one side. (Received January 14, 1953.)

291. Alfred Huber: *A theorem of Phragmén-Lindelöf type.*

For any real constant $k<1$ we consider solutions $u(x_1, x_2, \cdots, x_n)$ of $x_n\Delta u+ku_{x_n}=0$, defined in the half-space $H[x_n>0]$ and satisfying at the finite boundary B of H the condition $\lim\sup_{P\to Q} u(P)\leqq 0$ $(P\in H, Q\in B)$. Let S_r denote the half-sphere $H\cap[\sum_{i=1}^n x_i^2=r^2]$. The results are: (1) The limit $\alpha=\lim_{r\to\infty} m(r)/r^{1-k}$, where $m(r)=\sup_{P\in S_r} u(P)$, always exists (finite or infinite). (2) $\alpha\geqq 0$. (3) Throughout H the inequality $u\leqq\alpha x_n^{1-k}$ holds; if the equality is attained in at least one point of H, then $u\equiv\alpha x_n^{1-k}$. The theorem is an extension of the Phragmen-Lindelöf theorem for harmonic functions $(k=0)$ in the formulation of M. Heins (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 238–244). The work was done under the Office of Naval Research Contract No. N7onr-39705. (Received January 14, 1953.)

292. Benjamin Lepson: *On the variations of a function of bounded variation.*

It is well known that a necessary and sufficient condition that a real-valued function of a real variable be of bounded variation over a finite interval is that it can be expressed as the difference of two nondecreasing functions. The sufficiency is obvious, while the necessity is usually proved by decomposing the function into its positive and negative variations (aside from an additive constant). This may be called the *canonical decomposition* of the function. However, it is readily seen that, given a pair of nondecreasing functions, it is not always possible to find a function of bounded variation for which they form such a decomposition. It is the purpose of this paper to find conditions on the given pair of functions so that this is possible. (Received January 15, 1953.)

293*t*. A. E. Livingston: *The Lebesgue constants for Euler* (ϵ, t), $0<t<1$, *summation of Fourier series.*

The Lebesgue constant of order n for the Euler (ϵ, t), $0<t<1$, means of a trigonometric Fourier series is $L_n(t)=(2/\pi)\int_0^\pi|\text{Im}\,[e^{ix/2}(1-t+te^{ix})^n]|/(2\sin x/2)dx$. (The summability method (ϵ, t) is ordinarily denoted by $(E, (1-t)/t)$.) The author submitted an abstract at the New Haven meeting of October 25, 1952, in which he stated that $L_n(t)=(2/\pi^2)\log n+O(1)$ for each t as $n\to\infty$. He has now completed this result to $L_n(t)=(2/\pi^2)\log(2nt/(1-t))-(2/\pi^2)C+(2/\pi)\int_0^1 x^{-1}\sin x\,dx-(2/\pi)\int_1^\infty x^{-1}\{2/\pi-|\sin x|\}dx+\epsilon_n(t)$, where C is the Euler-Mascheroni constant and $\epsilon_n(t)\to 0$ for each t as $n\to\infty$. (Received February 25, 1953.)

294t. A. E. Livingston: *The zeros of a certain class of indefinite integrals.*

Let $F(x)=\int_x^\infty f(t)g(t)dt$, where $f(t)\geqq 0$ on $[0, 1)$, $f(t)\not\equiv 0$ on any $[a, b)\subset[0, 1)$, $f(t)\in L(0, 1)$, $f(t+n)=(-1)^n f(t)$, for $n=1, 2, \cdots$, $g(t)>0$ for $t\geqq 0$, $g(t)\downarrow 0$, $f(t)g(t)\in L(0, 1)$, $g(n+1)/g(n)\to 1$ as $n\to\infty$, and $g(t)$ has a point of decrease in each of an infinite set of intervals $(n, n+1)$, n integral. Let z_n denote the zero of $F(x)$ in $(n, n+1)$, $n=0, 1, \cdots$. If N is an infinite set of distinct non-negative integers and $\alpha\in[0, 1]$, then the author shows that a n.a.s.c. for $(-1)^n F(n)/g(n)\to\int_0^\alpha f(t)dt$ as $n\to\infty$, $n\in N$, is that $z_n-n\to\alpha$ as $n\to\infty$, $n\in N$. If, in addition, $g(t)-g(t+1)\in\downarrow$ and if $C\in(0, 1)$ is defined by $2\int_0^C f(t)dt=\int_0^1 f(t)dt$, then it is shown that $(-1)^n F(n)/g(n)\to\int_0^C f(t)dt$ as $n\to\infty$ and that $z_n-n\geqq C$, $n=0, 1, \cdots$. (Received February 24, 1953.)

295. Eugene Lukacs and Otto Szász: *Non-negative trigonometric polynomials and certain rational characteristic functions.*

Let $0<b_1<b_2<\cdots<b_n$ be n integers and $0<d_1<d_2<\cdots<d_m$ be m $(m\leqq n)$ real numbers (not necessarily integers). Denote by $g(\theta)$ the Vandermonde determinant formed from the $b_1^2, b_2^2, \cdots, b_n^2$ with the first row replaced by $1-\lambda_j\cos b_j\theta$ where $\lambda_j=\prod_{k=1}^m(1-b_j^2/d_k^2)$ for $j=1, 2, \cdots, n$. The question whether $g(\theta)$ is non-negative for all values of θ is closely connected with the problem whether certain rational functions are characteristic functions. Four configurations of the $b_1, \cdots, b_n$ and $d_1, \cdots, d_m$ are are studied which lead to non-negative trigonometric polynomials. (Received January 8, 1953.)

296. M. H. Protter: *A boundary value problem for the wave equation.*

Let x_0, y_0, z_0 be fixed quantities such that $x_0^2+y_0^2<z_0^2$ and suppose D is the domain in (x, y, z) space bounded by the surfaces S_1: $x^2+y^2=(z-z_0)^2$, S_2: $(x-x_0)^2+(y-y_0)^2=z^2$, and S_3: $z=0$. Let $f(x, y)$, $g(x, y)$ be given functions having continuous first derivatives. A solution to the wave equation $u_{xx}+u_{yy}=u_{zz}$ is obtained in D satisfying the boundary conditions $u=f(x, y)$ on S_3, $u=g(x, y)$ on S_2. The proof uses the generalization of the Riemann method due to Martin as well as Asgeirsson's mean value theorem. The result can be extended to the wave equation in any number of variables. As a corollary certain theorems on mean values of a function defined in a circle may be obtained. (Received October 22, 1952.)

297. I. F. Ritter: *A parametric representation of the eigenvectors of a matrix.*

The eigenvalue problem for a matrix A of order n with numerical elements can be formulated by the identity (1) $(pA-I)\bar{y}(p)\equiv f(p)\bar{c}$ in the parameter p, where $f(\lambda^{-1})\lambda^n$ is the characteristic function of A and $\bar{c}$ a numerical vector. If $f(p)$ and a vector $\bar{y}(p)$, satisfying (1), are obtained in polynomial form, each root p_i of $f=0$ furnishes an eigenvector $\bar{y}(p_i)$ of A. A practical method, which remains feasible with increasing n, is described for computing, before f is found by (1), a solution $\bar{y}(p)$ in a form which differs significantly from the columns of adj $(pA-I)$. Beginning with a vector $\bar{r}^1$ with n arbitrary parameters as components, a sequence $\bar{r}^i(p)$ is set up by eliminating from $pA\bar{r}^{i-1}(p)$ one of the parameters to form $\bar{r}^i(p)$. For nonderogatory A, with trivial exceptions, $\bar{r}^n(p)$, containing one arbitrary parameter, can be reached and taken as solution $\bar{y}(p)$. If A is derogatory, the sequence cannot be continued beyond $\bar{z}=\bar{r}^k(p)$, $k<n$. Then one component of the vector $\bar{g}(p)=(pA-I)\bar{z}(p)$ represents the reduced characteristic function $g(\lambda^{-1})\lambda^k$ of A. After solving $g=0$ (much easier to solve than $f=0$) for the eigenvalues p_i of A, either all (except one) of the parameters in $\bar{z}(p_i)$ can be determined from $p_iA\bar{z}(p_i)=\bar{z}(p_i)$, or enough of them remain arbitrary so that $\bar{z}(p_i)$ represents the invariant subspace belonging to the eigenvalue p_i. (Received January 14, 1953.)

298*t*. M. S. Robertson: *Schlicht solutions of the Sturm-Liouville differential equation*. Preliminary report.

Let $zp(z)$ be regular for $|z|<1$ and $\Re\{z^2p(z)\}\leqq(x_1^2/4)|z|$ for $|z|<1$, where X_1 is the smallest positive zero of the Bessel function $J_0(x)$. Then the unique solution $W=W(z)$ of the differential equation $W''+p(z)W=0$ for which $W(0)=0$, $W'(0)=1$, is schlicht and star-like in $|z|<1$. The constant $x_1^2/4=1.4460\cdots$ is a best possible one. A similar theorem holds if $\Re\{z^2p(z)\}\leqq(\pi^2/4)|z|^2$ in $|z|<1$, and again the constant $(\pi^2/4)$ is a best possible one. If $z^2p(z)$ is regular in $|z|<1$, and if $\Re\{z^2p(z)\}\leqq K$, where $K=S_1^2/2+1/8$ and S_1 is the smallest positive zero $S=S_1$ of the function $\phi(S)=\int_0^\infty f(u)\cos su\,du$, $f(u)=(3+\cosh u)^{-1/2}$, then the unique solution of the form $W=W(z)=z^\alpha\sum_{n=1}^\infty a_nz^n$, $a_0=1$, of $W''+p(z)W=0$, corresponding to the root α of the indicial equation whose real part is the larger, is such that $\{W(z)\}^{1/\alpha}=z+\cdots$ is schlicht for $|z|<1$. The constant K cannot be replaced by a larger one. These theorems are special cases of a more general parent theorem which will be announced later on completion of the present investigation. (Received January 12, 1953.)

299*t*. L. B. Robinson: *On a complete system of tensors.*

Given the system of equations $-y_j\partial f/\partial y_i-y_j'\,\partial f/\partial y_i'-y_j''\,\partial f/\partial y_i''-I_j\partial f/\partial I_i=0$, $i\neq j$; $i, j=1, 2, 3$, $-y_1\partial f/\partial y_1-y_1'\,\partial f/\partial y_1'-y_1''\,\partial f/\partial y_1''+I_2\partial f/\partial I_2+I_3\partial f/\partial I_3+I_4\partial f/\partial I_4=0$, $-y_2\partial f/\partial y_2-y_2'\,\partial f/y_2'-y_2''\,\partial f/\partial y_2''+I_1\partial f/\partial I_1+I_3\partial f/\partial I_3+I_4\partial f/\partial I_4=0$, $-y_3\partial f/\partial y_3-y_3'\,\partial f/\partial y_3'-y_3''\,\partial f/\partial y_3''+I_1\partial f/\partial I_1+I_2\partial f/\partial I_2+I_4\partial f/\partial I_4=0$, $-\Theta\partial f/\partial\Theta-y_1'\,\partial f/\partial y_1'-y_2'\,\partial f/\partial y_2'-y_3'\,\partial f/\partial y_3'+I_1\partial f/\partial I_1+I_2\partial f/\partial I_2+I_3\partial f/\partial I_3=0$. Solving these equations helps one to compute a complete system of tensors. (Received January 5, 1953.)

300. Maxwell Rosenlicht: *Simple differentials of second kind on Hodge manifolds.*

It is proved that on a Hodge manifold (in particular, on a nonsingular algebraic variety), the number of simple differentials of second kind that are linearly independent modulo exact differentials is precisely the first Betti number. The theory of abelian varieties is used to effect a simple reduction of the proof of the general theorem to the case of an algebraic curve. (Received January 12, 1953.)

301. V. L. Shapiro: *Circular summability C of double trigonometric series.*

$F(x, y)$, defined in a neighborhood of (x_0, y_0) and integrable on the circumference of every circle contained in this neighborhood with (x_0, y_0) as center, is said to have a generalized rth Laplacian at (x_0, y_0) equal to a_r if $(1/2\pi)\int_0^{2\pi}F(x_0+t\cos\theta, y_0+t\sin\theta)d\theta$

$=a_0+a_1t^2/[2!]^2+\cdots+a_rt^{2r}/[2^rr!]^2+o(t^{2r})$. By means of this definition a necessary and sufficient condition that there exist a Cesaro mean which sums a given series is obtained. In particular the following theorem is proved: Let $T=\sum c_{mn}e^{i(mx+ny)}$ be a double trigonometric series with coefficients $c_{mn}=o((m^2+n^2)^\gamma)$, $\gamma \geqq -1$. A necessary and sufficient condition that the series should be circularly summable C at the point (x_0, y_0) to the sum s is that there exists an integer $r>\gamma+1$ such that if $F(x, y)=c_{00}(x+y)^{2r}/2^r[(2r)!]+\sum_{|M|\neq 0}(-1)^rc_{mn}e^{i(mx+ny)}/(m^2+n^2)^r$, then the generalized rth Laplacian of F exists at the point (x_0, y_0) and is equal to s. (Received January 2, 1953.)

302*t*. J. M. Stark: *Distortion of analytic angle in pseudo-conformal transformation.*

Using Bergman's method of the minimum integral (Mémorial des Sciences Mathématiques, vol. 106, p. 40), the author gives bounds for the distortion of the analytic angle A (Ibid., p. 9) between two directions at a point $P\in B$ under pseudo-conformal transformations of four-dimensional domain B. Such bounds result from bounds for the ratios between the euclidean $E(A)$ and non-euclidean $N(A; B)$ measures of A. $N(A; B)$ is obtained using Bergman's metric defined in B and invariant with respect to pseudo-conformal transformations (Ibid., p. 51). $E(A)/N(A; B)$ is shown to depend upon euclidean measures of B obtained using different weighting functions. E.g., $c(m/M)^{10}\leqq \sin E(A)/\sin N(A; B)\leqq d(M/m)^{10}$ where m is the distance of P from the boundary b of B, M is the maximum distance of a point (Z_1, Z_2) from b, c and d are simple functions of $|Z_1|/M$, $|Z_2|/M$, W/M^4, W/m^4 where W is a constant $\geqq$ euclidean volume of B. The author obtains other bounds for distortion using the first few functions of an orthonormal set in B and various euclidean measures of B with conveniently chosen weighting functions. (Received February 2, 1953.)

303*t*. J. M. Stark: *Distortion theorems for pseudo-conformal transformations.*

The author derives relations connecting the minima λ_B of the integral $\int_B|f|^2d\omega$, $d\omega=dx_1dy_1dx_2dy_2$, $z_k=x_k+iy_k$, $k=1, 2$, for functions $f=f(z_1, z_2)\in L^2(B)$ and satisfying at a point $t=(t_1, t_2)\in B$ certain auxiliary conditions. E.g., if (a) $f(t)=1$, (b) $f(t)=0$, (c) $(\partial f/\partial z_1)_t=1$, (d) $\int_B f d\omega=0$ are auxiliary conditions, let λ_B^1, λ_B^{*1}, λ_B^{01}, λ_{BB}^{01} be the minima of $\int_B|f|^2d\omega$ under conditions $\{(a)\}$, $\{(c)\}$, $\{(b)\ (c)\}$, $\{(b)\ (c)\ (d)\}$, respectively. Then $(1/\lambda_B^{01})=(1/\lambda_{BB}^{01})+(\lambda_B^1/\mathrm{Vol}\ B)\{1/\lambda_B^{*1}-1/\lambda_{BB}^{01}\}$ where Vol B is the euclidean volume of B. Using Bergman's method of the minimum integral (Mémorial des Sciences Mathématiques, 106, p. 40 and vol. 108, p. 48) and assuming known some quantities characteristic of B, such as an upper bound for Vol B, euclidean measures of B with certain weighting functions, etc., inequalities are obtained for various λ_B which are then used in distortion theorems. Bounds for distortion of arc length in an arbitrary direction as well as bounds which depend to some extent upon the direction are derived. (Received February 2, 1953.)

304. J. L. Walsh and D. M. Young, Jr.: *Degree of convergence of solution of difference equation to solution of Dirichlet problem.*

For the unit square with square grid of mesh $1/A$, in any closed interior region the convergence of the solution of the difference equation to the solution of the Dirichlet problem is of the respective orders $1/A^2$, $1/A$, $1/A^\alpha$, provided the boundary values either have a derivative of bounded variation, or are themselves of bounded variation, or satisfy a Lipschitz condition of order α, $0<\alpha\leqq 1$. For boundary values piecewise constant the convergence is of the order $1/A^2$ or $1/A$ according as all discontinuities are or are not points of the grid. If only continuity is assumed for the boundary values, the order of the convergence may be arbitrarily slow with respect to $1/A$. (Received January 12, 1953.)

305t. John Wermer: *On a class of bounded linear operators*. Preliminary report.

Let B be a Banach space, S a bounded operator on B with inverse, D that component of the resolvent set of S which contains the origin. Choose x in B, set C equal to the smallest closed subspace of B invariant under S and containing x, and suppose $S^{-1}x$ not in C. Let T denote the restriction of S to C. *Theorem: For each f in C there is a unique function $F(z)$ analytic in D such that for $n=0, 1, \cdots : f=F(0)x+F'(0)Tx+(F''(0)/2!)T^2x+\cdots+(F^{(n)}(0)/n!)T^nx+T^{n+1}y_n$, y_n in C.* Let R denote the ring of operators U, defined on C, with $UT=TU$. *Theorem: If for no $y\neq 0$ in C, $T^{-n}y$ is in C for all n, then R is isomorphic to a ring of functions analytic on D. In particular, if E is in R, $E^2=E$, then $E=I$ or $E=0$.* Certain special operators are discussed in detail as examples of the two theorems. (Received January 15, 1953.)

306. John Wermer: *On algebras of continuous functions*.

Let C denote the algebra of functions continuous on the unit circle. With the norm: $\|f\|=\sup_{|\lambda|=1}|f(\lambda)|$, C is a Banach algebra. Let A denote the set of all f in C which are boundary values of functions analytic in $|z|<1$ and continuous in $|z|\leqq 1$. A is then a closed subalgebra of C, and by known results A consists precisely of those f in C for which $\int_{|\lambda|=1}f(\lambda)\lambda^n d\lambda=0$, $n\geqq 0$. The following theorem is proved: If A' is any closed subalgebra of C which includes A, then either $A'=A$ or $A'=C$. (Received January 9, 1953.)

307. Albert Wilansky: *Row-finite and row-infinite summability matrices*.

In Proc. Amer. Math. Soc. vol. 3 (1952) p. 389, the author proved that a reversible matrix allowed an equipotent consistent row-infinite matrix, thus in particular, a stronger, consistent row-infinite matrix, and conjectured that the latter is true with normal (=reversible-triangular) instead of row-infinite. The conjecture is correct for A if (1) there exists R_n with $\lim_n \sum_{k>R_n} a_{nk}x_k=0$ for all x summable A, (2) $\sum_k |a_{nk}/t_k|<\infty$ for each n, where t_n (known to exist, loc. cit. p. 390) satisfies $\sup|t_nx_n|<\infty$ for each x summable A. *Theorem.* There is a normal matrix consistent and equipotent with any reversible row-finite matrix. The theorem is trivial if "triangular" is substituted for "normal." (Received January 7, 1953.)

Applied Mathematics

308. Wallace Givens: *Numerical computation of the characteristic values of a real symmetric matrix*.

The problem considered is that of finding the characteristic values of a real symmetric matrix $\bar{A}$ by a method suitable for use with a high speed automatic sequenced digital computer. Let $\bar{A}$ have digital elements and norm slightly less than one; $\bar{A}$ is not required to be definite nor need its roots be separated. By a sequence of $(n-1)(n-2)/2$ rotations in the coordinate planes x_2x_3, x_2x_4, $\cdots$, x_2x_n, x_3x_4, $\cdots$,

$x_3x_n, \cdots, x_{n-1}x_n$, the quadratic form $x'\bar{A}x$ is carried into $x'\bar{S}x$ with $\bar{S} = (\bar{s}_{ij})$ and $\bar{s}_{ij}=0$ for $|i-j|>1$. After this first stage of "concentrating the data," the roots are found by determining the signature of $\bar{S}-\bar{\mu}I_n$ for suitable values of $\bar{\mu}$. This general mathematical procedure is described in a paper to appear in the Proceedings of an NBS Symposium held in Los Angeles, August 1951 [NBS Appl. Math. Ser., vol. 29]. The present paper is a detailed error analysis, using methods stemming from those of von Neumann and Goldstine [Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1021–1099], which proves the stability of the computation and establishes satisfactory bounds for the error (produced by round-off) in any root. For the second stage the bound is particularly small and is independent of the order of the matrix. (Received January 14, 1953.)

309. Imanuel Marx: *Recurrence relations for prolate spheroidal wave functions.*

Solution of the three-dimensional wave equation by separation in prolate spheroidal coordinates ξ, η, ϕ leads to the angular wave functions $S_{mn}(\eta)$ and the radial wave functions $R_{mn}(\xi)$, defined over the ranges $-1\leqq\eta\leqq 1$ and $1\leqq\xi<\infty$, respectively. Only functions of the first kind, which have no logarithmic terms, are considered. The functions $\psi_{mn}=S_{mn}R_{mn}\cos m\phi$ and $\omega_{mn}=S_{mn}R_{mn}\sin m\phi$ solve the wave equation, and an arbitrary regular solution may be expanded in a termwise integrable double series of these functions. The derivatives $\partial\psi_{mn}/\partial x$, $\partial\psi_{mn}/\partial y$, $\partial\psi_{mn}/\partial z$ are likewise solutions. The trigonometric as well as the wave functions have integral orthogonality over their respective intervals. Multiplication of $\partial\psi_{mn}/\partial z$ by $\cos p\phi$ and integration with respect to ϕ, followed by multiplication by S_{pq} and integration with respect to η, gives a linear relation among R_{mn}, R'_{mn}, and R_{mq}, $q\neq n$. Multiplication by R_{pq} and integration with respect to ξ similarly gives a linear relation among S_{mn}, S'_{mn}, and S_{mq}. The same method applied to $\partial\psi_{mn}/\partial x$ and $\partial\psi_{mn}/\partial y$ gives relations among R_{mn}, R'_{mn}, and $R_{m\pm 1,q}$, or among S_{mn}, S'_{mn}, and $S_{m\pm 1,q}$. (Received January 12, 1953.)

310*t*. L. E. Payne and H. F. Weinberger: *Upper and lower bounds for polarization.*

The following problem is considered. $\Delta\phi=0$ in the exterior of a closed surface S. ϕ assumes the boundary values x_i (a rectangular coordinate) on S and vanishes at infinity. The Dirichlet integral of ϕ is called the polarization P of S. Certain theorems concerning the value of P_{ii} are proved. Among these are: (a) $P_{11}+P_{22}+P_{33}\geqq 6V_s$, where V_s is the volume of S. This inequality was conjectured by Schiffer and Szegö (Trans. Amer. Math. Soc. vol. 67 (1949)) and becomes an equality when S is a sphere. (b) If S is cut in two by a plane perpendicular to the x_i-axis and the two half-bodies so resulting are reflected in the plane to form two symmetric bodies A and B, then $P_{ii}(S)\leqq[P_{ii}(A)+P_{ii}(B)]/2$ and $P_{jj}(S)\geqq[P_{jj}(A)+P_{jj}(B)]/2$ for $j\neq i$. (c) If S is symmetric about a plane perpendicular to the x_i-axis, then $P_{ii}\geqq(4\pi)^{2/5}(15I_{ii})^{3/5}-V_s$ where I_{ii} is the moment of inertia of S about the plane of symmetry. (d) If S is symmetric about a plane perpendicular to the x_i-axis, $P_{ii}\leqq[\int_0^\infty dv/F(v)]^{-1}-V_s$ where $F(v)=M+M_1v+M_2v^2+M_3v^3+(4\pi/3)v^4$ and M_i are the analogues in 5 dimensions of the Minkowski number of the convex hull of S. This work was sponsored by the Office of Naval Research. (Received January 16, 1953.)

311. E. H. Zarantonello: *Singularities on free boundaries.*

The free boundaries of a plane, steady irrotational flow of an ideal fluid may carry isolated singularities. Well known are the singularities generating jets, wakes, and planning surfaces (cf. Milne-Thomson, *Theoretical hydrodynamics*, Chap. 12). An investigation of all possible singularities, under the only assumptions of a bounded velocity and a locally schlicht flow, leads essentially to these three types only. For this purpose the flow is locally mapped onto a domain of a parameter plane t with the free boundary corresponding to a segment of the real axis and the singularity to $t=0$. Then, at the origin, the complex potential w and the complex velocity ζ can have only the following expansions: $w=at^{-2}+bt^{-1}+c \ln t +R_1(t)$, $\zeta(t)=t^{im} \exp\{iR_2(t)\}$, where $R_1(t)$ and $R_2(t)$ are power series of t, real on the real axis. $m=0$ corresponds to the above-mentioned singularities and $m\neq 0$ yields four new types in which one free streamline curls around another of different velocity. These results are obtained by a combination of Koebe's distortion theorem and a result due to Stone characterizing poles from the behavior of the functions with regard to the distance to a straight line through the pole (M. H. Stone, *On a theorem of Pólya*, J. Indian Math. Soc. N.S. vol. 12 (1948) pp. 1–7). This work has been sponsored by the Office of Naval Research. (Received January 15, 1953.)

Geometry

312. L. W. Green: *Surfaces without conjugate points.*

Let M be a two-dimensional simply-connected manifold, complete under a Riemannian metric of class C^3. Assume that the Gaussian curvature of M is bounded below. A pole of M is a point Q such that any two complete geodesic rays with initial point Q intersect only at Q. It is proved that, if P is interior to an open set of poles of M, then any two geodesic rays with initial point P must diverge. In particular, this holds for any intersecting geodesic rays if every point of M is a pole, i.e., if no geodesic of M has a pair of mutually conjugate points. This implies the existence of topological transitivity of the geodesic flow on a class of surfaces for which M is the universal covering surface, including those treated by Morse and Hedlund [Trans. Amer. Math. Soc. vol. 51 (1942) pp. 362–386]. If, in addition, it is assumed that there are no points focal to any geodesic of M, it is shown that the Gaussian curvature between two "parallel" geodesics must vanish. This implies that a surface homeomorphic to a cylinder which has bounded opening and no focal points must have zero curvature. (Received January 15, 1953.)

Logic and Foundations

313*t*. E. L. Post: *Solvability, definability, provability; history of an error.*

The concept general recursive function leads to a thesis (Kleene, Trans. Amer. Math. Soc. (1943) p. 60) having as a consequence the existence of absolutely unsolvable problems. Also in 1943, the author (Amer. J. Math., footnote 6) proposed the problem of absolutely undecidable propositions. The author later realized that a necessary condition for provability would suffice; still later, that to solvability and provability should be added definability. In 1947, in a lost letter to Tarski via Church, the author proposed a formulation of "Primitive Inductive-Reflective Proof" involving, besides the elementary, only mathematical induction and "Gödelization." But a representation theory for Gödel's system P (1931) failed to materialize due to P's Axiom of Reducibility. Indeed, Kleene's Example (Proceedings of the International Congress of Mathematicians, 1950, vol. II, p. 683) offers hope of an impossibility proof. The obvious specificity of Mostowski's "definable set" concept (Fund. Math. (1947)) suggested a like development for the language (set of W. F. F.) of system P, and, constructivity being lost, further led to that generalization which constitutes the language of the P^α of the author's present research (see following abstract). Just prior to the last idea it was seen that the correct order of the earlier triplet is, of course, solvability, definability, provability. (Received February 9, 1953.)

314t. E. L. Post: *A necessary condition for definability for transfinite von Neumann-Gödel set theory sets, with an application to the problem of the existence of a definable well-ordering of the continuum.* Preliminary report.

In $P^{(\alpha)}$ (see previous abstract), α is an arbitrary ordinal. The language (sic) of $P^{(\alpha)}$ is that of the simple theory of types semantically treated. An "α-formula" is defined as an $x_i^{(\alpha)}$ ($x_j^{(\alpha)}$) range, all sets of type less than α, and any formula obtainable from α-formulas by a negation, disjunction or quantification. The present thesis is that every definable set is given by some α-formula. The well-ordered series of ordinals leads to a well-ordering A of the continuum—represented by a set of positive integers in classic manner. A Gödelian interval $0 \leqq \xi < 1$, ξ sequence of definitions proves that A is an "α-set" and, subject to the filling in of five validation lacunae and checking, by a calculus of validation, the other formulas, leads to the theorem (the corresponding metatheorem is obvious): either (the defined) A well-orders the continuum-interval $0 \leqq \xi < 1$ or there is no definable well-ordering of the continuum. Further verifications of the thesis separate themselves into theorems: A. That certain (a) generalizations, (b) modifications, of $P^{(\alpha)}$ lead only to α-sets. B. That a certain diagonal set of a denumerable set under certain (a) general, (b) particular conditions is again an α-set. (Received February 9, 1953.)

Statistics and Probability

315. A. G. Anderson: *The prediction of quantitative characteristics in polygenic systems.* I.

A method is devised whereby the expected mean value of a given quantitative characteristic of a hybridized strain descended in any manner from members of some original set of inbred strains can be predicted. This is done by showing the system having as elements the representations of various stocks in terms of their proportions of gene-pairs involving two allelic genes which contribute equally and additively to the quantitative characteristic under consideration to be isomorphic to a subset of the normalized elements having non-negative coefficients in a nonassociative baric algebra of order three. The expected proportions of loci of the three possible types are thus obtainable as functions of the proportions in the parent stocks, and it is shown that the proportions of loci carrying pairs of the effective gene can be estimated for the members of a set of inbred stocks if the values of the quantitative characteristic are known for the stocks themselves and the F_1 hybrids resulting from crosses among them. The crossing of related stocks is discussed and it is made clear that such crosses do not lead to the most desirable values of the quantitative characteristic in question. (Received January 16, 1953.)

316t. K. L. Chung: *Note on unimodal distribution.*

A distribution function $F(x)$ is called unimodal with vertex s if $F(x)$ is convex for $x<s$, and concave for $x>s$. A. I. Lapin claimed in his thesis (1947, in Russian and unavailable in the U.S.) that the convolution of two distribution functions both unimodal with vertex 0 is again so. Using this theorem Gnedenko (Ukrain. Mat. Zurnal vol. 1 (1949) pp. 3–8) proved that every d.f. belonging to the class L is unimodal. Lapin's theorem (and proof) is trivially incorrect if one considers the uniform distribution in the interval $(-1/3, 2/3)$. Even without specifying the vertex of the convolution the theorem is still incorrect. Example. $F(x)=0$ if $x<-c$, $=1/2+x/2c$ if $-c\leqq x<0$, $=1/2+x$ if $0\leqq x<1/2$, $=1$ if $1/2\leqq x$. For any $c<1/4$ the convolution of F with itself is not unimodal, in fact its derivative has two relative maxima. Thus Gnedenko's theorem remains unproven. (Received January 28, 1953.)

317. R. E. Kalaba: *A random walk interpretation for the Poisson, Pólya, and Neyman distributions.*

A generalization of the well known Poisson random walk process is considered, in that steps of arbitrary integral length and direction along a line at each instant are permitted. The probability of being at a position n steps from the initial point after time t is obtained, as are certain general properties of the distribution. It is shown that the "contagious" distributions of Pólya and Neyman (as well as the Poisson and other distributions) are obtained as special cases, thus bringing out sharply points of difference and similarity among these distributions; by way of illustration, an application to fluctuation phenomena of high energy particles is pointed out. The point of view taken in the paper enables one to generalize results contained in the less well known papers of Lüders, Cernuschi, and Castagnetto, and to correlate them with Neyman's, Pólya's, and especially Feller's papers on contagion. (Received January 7, 1953.)

318t. G. Kallianpur and Herbert Robbins: *On the equidistribution in probability of sums of independent random variables.*

Let $X_1, X_2, \cdots$ be independent random variables with a common distribution function $F(x)$, let $S_n=X_1+\cdots+X_n$, and let $U_n(h_1, h_2)=\sum_{j=1}^{n} h_1(S_j)/\sum_{j=1}^{n} h_2(S_j)$. Under mild restrictions on $F(x)$ (for example, if (i) $F'(x)$ exists and belongs to $L^p(-\infty, \infty)$ for some $1<p\leqq 2$, and (ii) either the mean of $F(x)$ is 0 and the second moment finite, or Pr $[n^{-1/\alpha}S_n\leqq x]\rightarrow V_\alpha(x)=$ symmetric stable law with characteristic exponent $1\leqq\alpha<2$) it is shown that the sequence $\{S_n\}$ is *equidistributed* in the sense that if each $h_i(x)$ is bounded and summable on $(-\infty, \infty)$ with integral $\bar{h}_i$, $\bar{h}_2\neq 0$, then $\lim_{n\rightarrow\infty} U_n=\bar{h}_1/\bar{h}_2$ in probability. The same result holds if the $h_i(x)$ are Riemann integrable in some finite interval, vanishing elsewhere, even when $F(x)$ is not absolutely continuous, provided it is not a lattice distribution and has mean 0 and finite third absolute moment. For a large class of $F(x)$ belonging to the domain of normal attraction of the symmetric Cauchy law this property is preserved under translation. On the other hand, if the mean of $F(x)$ is $\neq 0$ then $\{S_n\}$ is not equidistributed. As an incidental result the limiting distribution of sums $\sum_1^n h(S_j)$, properly normalized, is found in the case of equidistribution, generalizing results of Chung and Kac (*Remarks on fluctuations of sums of independent random variables*, Memoirs of the American Mathematical Society, no. 6, 1951). (Received January 23, 1953.)

Topology

319. R. D. Anderson: *Some monotone (and monotone open) images of R^3.*

Let R^p denote the p-dimensional cube. The author shows that for any positive integer p, there exists a monotone mapping of R^3 onto R^p. With the previously announced result of the author to the effect that if there is a monotone mapping g of a compact n-manifold M (with or without boundary) ($n \geqq 3$), then there is a monotone open mapping of M onto $g(M)$, this establishes the theorem that for any positive integer p, there is a monotone open mapping of R^3 onto R^p. (Received January 13, 1953.)

320. S. I. Goldberg: *On the "Euler characteristic" of a semi-simple Lie algebra.*

Let L be a semi-simple Lie algebra of dimension n over a field of characteristic 0 and denote the qth Betti number of L by R^q. By definition $\sum_{q=0}^{n}(-1)^q R^q$ is the Euler characteristic $\chi(L)$ of L. From transcendental properties of compact Lie groups (Hopf's theorem and Weyl's unitary trick) it has been shown that $\chi(L)=0$ (C. Chevalley and S. Eilenberg, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 85–124). In this paper an elementary algebraic proof of this result is given. (Received January 16, 1953.)

321t. V. L. Klee, Jr.: *Some spaces which lack the fixed-point property.*

This note complements some well known fixed-point theorems by showing that certain classes of metric spaces do *not* have the fixed-point property. Specifically, if X is a noncompact metric space which is either (a) connected, locally connected, and locally compact, or (b) a convex subset of a normed linear space, then X contains a topological ray T as a closed subset. But then, since X must admit a retraction onto T one obtains (1) A locally compact connected ANR X is compact if and only if every null-homotopic map of X into X admits at least one fixed-point; (2) A convex subset of a normed linear space is compact if and only if it has the fixed-point property. (Received January 8, 1953.)

322t. E. A. Michael: *On a problem of S. Kaplan.*

An abelian topological group is called a P-group if the natural homomorphism from G into its second character group is a homeomorphism onto. S. Kaplan asked [Duke Math. J. vol. 17 (1950) pp. 419–435] whether every inverse limit of P-groups is a P-group. The answer is "no." Since M. Smith has proved [Ann. of Math. vol. 53 (1952) pp. 248–253] that every Banach space is a P-group, and the author has proved [Memoirs of the American Mathematical Society, no. 11, 1952] that every complete locally convex topological linear space is an inverse limit of Banach spaces, it is sufficient to give an example of a complete locally convex space which is not a P-group. Such an example is provided by the space $C(I)$ of continuous, real-valued functions on the closed unit interval I, in the topology of uniform convergence on compact-and-countable subsets of I. The completeness of $C(I)$ was established in the above paper of the author; that $C(I)$ is not a P-group follows from a criterion in the above paper of M. Smith, and the fact that I, considered as a set of continuous linear functionals on $C(I)$, is compact in the compact-open topology but not equicontinuous. (Received January 16, 1953.)

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