ON DESCARTES-HARRIOT'S RULE[1]

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The classical rule of Descartes-Harriot asserts that if

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \tag{1}$$

is a polynomial with real coefficients, then the number P of its positive zeros cannot exceed the number V_0 of the sign changes of the sequence

$$a_0, a_1, \cdots, a_n \tag{2}$$

(if some of the coefficients vanish they can be omitted). This rule has been extended and refined in various ways. V_0 yields obviously an upper bound for the number P^* of zeros lying in $0 < x < 1$; then Laguerre[2] observed the P^* is majorised also by the number V_1 of sign changes of the sequence

$$a_0, (a_0 + a_1), \cdots, (a_0 + a_1 + \cdots + a_n), (a_0 + \cdots + a_n), \cdots \tag{3}$$

which is not greater than V_0. More generally P^* is majorised by V_k, where V_k denotes the number of sign changes of that sequence which arises from (2), completing it with zeros to an infinite sequence and forming the sequence of k times iterated partial sums; these V_k's form a nonincreasing sequence. One trend of the investigations is the study of the V_k for large k; this was done mainly by M. Fekete and G. Pólya.[3] Another trend is to obtain similar rules representing $f(x)$ in various other forms (Runge,[4] Sylvester,[5] Obreschkoff,[6] I. J. Schoenberg[7]) or even generally for the linear combinations

$$h(x) = a_0\phi_0(x) + \cdots + a_n\phi_n(x)$$

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[1] To the memory of my late friend, Ervin Feldheim.

[2] E. Laguerre, *Mémoire sur la théorie des équations numériques*, J. Math. Pures Appl. (3) vol. 9 (1883) pp. 99–146.

[3] See in particular their joint paper in Rend. Circ. Mat. Palermo vol. 34 (1912) pp. 89–120 entitled *Über ein Problem von Laguerre.*

[4] See the paper of G. Pólya, *Über einige Verallgemeinerungen der Descartesschen Zeichenregel*, Archiv der Mathematik und Physik vol. 23 (1915) pp. 22–32.

[5] J. J. Sylvester, *Mathematical papers*, vol. 2, pp. 360 and 401.

[6] N. Obreschkoff, *Über die Wurzeln algebraischer Gleichungen*, J. Deutschen Math. Verein vol. 33 (1924) pp. 52–64.

[7] I. J. Schoenberg, *Zur Abzählung der reellen Wurzeln algebraischen Gleichungen*, Math. Zeit. vol. 38 (1934) pp. 546–564.

of a prescribed system $\phi_0(x), \phi_1(x), \cdots$ (Pólya-Szegö,[8] S. Bernstein[9]). A further trend is the comparison of the different rules (F. Klein,[10] I. J. Schoenberg[7]), extensions permitting complex coefficients in (1) and substituting the segments of the positive axis by more general domains (Obreschkoff,[11] I. J. Schoenberg[12]), and finally the improving the rule by suitable Tschirnhausen-transformation (S. Lipka[13]). But as far as we know no similar rule has been given to a *lower* estimation of the number P of the positive roots. The aim of this note is to give such a simple rule. This is based on the Laguerre-polynomial-representation of $f(x)$:

$$f(x) = \sum_{\nu=0}^{n} b_\nu L_\nu(x), \qquad b_\nu \text{ real}, \tag{4}$$

where the Laguerre-polynomials $L_\nu(x)$ are defined as is well known by

$$e^{-x}L_\nu(x) = \frac{1}{\nu!}\frac{d^\nu}{dx^\nu}(e^{-x}x^\nu). \tag{5}$$

The rule we are going to prove will assert that *the number P of positive roots of $f(x)$ is not less than the number of sign changes in the sequence of successive differences of the coefficients b_ν, that is, in the sequence*

$$\begin{gathered} b_0,\ [b_0 - b_1],\ [b_0 - 2b_1 + b_2], \cdots, \\ [b_0 - C_{n,1}b_1 + C_{n,2}b_2 - \cdots + (-1)^n C_{n,n} b_n]. \end{gathered} \tag{6}$$

It is perhaps not uninteresting to notice that if η_n denotes the greatest positive root of $L_n(x)$, then[6] the number of sign changes of the coefficient-sequence

$$b_0, b_1, \cdots, b_n$$

itself furnishes an *upper* bound for the number of the roots of $f(x)$ in

[8] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, chap. 5, probleme 87–90.

[9] S. Bernstein, *Leçons sur les propriétés extremales et la meilleure approximation des fonctions analytiques d'une variable reelle*, Paris, 1926.

[10] F. Klein, *Geometrisches zur Abzählung der reellen Wurzeln algebraischer Gleichungen*, Gesammelte Mathematische Abhandlungen, pp. 198–208.

[11] N. Obreschkoff, *Sur un problème de Laguerre*, C. R. Acad. Sci. Paris vol. 177 (1923) pp. 102–104.

[12] I. J. Schoenberg, *Extensions of theorems of Descartes and Laguerre to the complex domain*, Duke Math. J. vol. 2 (1936) pp. 84–94.

[13] S. Lipka, *Über die Descartessche Zeichenregel*, Acta Univ. Szeged. vol. 7 (1934–1935) pp. 177–185.

the interval $\eta_n \leqq x < +\infty$.

The proof of the rule (6) is extremely simple and actually gives more since it turns out, as a matter of fact, that even the number of the positive sign changing places of $f(x)$ is not less than the number of sign changes of the sequence (6). It is based on the following theorem of Fejér.[14]

The number of the sign variations of a real function $g(x)$ in the interval $0 < x < a$ $(a \leqq +\infty)$ is not less than the number of sign changes of the sequence of moments:

$$M_0 = \int_0^a g(t)dt, \cdots, M_n = \int_0^a g(t)t^n dt. \tag{7}$$

In order to prove our rule we form instead of the moments of $f(x)$ those of $f(x)e^{-x}$

$$M_k = \int_0^\infty f(x)e^{-x}x^k dx, \qquad k = 0, 1, \cdots, n. \tag{8}$$

Owing to the expansion[15]

$$x^k = k! \sum_{\nu=0}^{k} (-1)^\nu C_{k,\nu} L_\nu(x) \tag{9}$$

and the well known orthogonality property[16]

$$\int_0^\infty e^{-x} L_\mu(x) L_\nu(x) dx = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \phantom{\text{if }} \mu = \nu, \end{cases}$$

we obtain from (9)

$$M_k = k! \sum_{\nu=0}^{k} (-1)^\nu C_{k,\nu} b_\nu, \qquad k = 0, 1, \cdots, n. \tag{10}$$

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[14] L. Fejér, *Nombre des changement de signe d'une fonction dans un intervalle et ses moments*, C. R. Acad. Sci. Paris vol. 158 (1914) pp. 1328–1331. If some of the moments are 0, then we can of course simply drop them; we can get an improved lower estimation in the following way. Let, for example, $M_\mu \neq 0$. Let $c_\mu = M_\mu$; further for $\nu < \mu$, $c_\nu = M_\nu$ if $M_\nu \neq 0$ and $c_\nu = -\text{sg}\, c_{\nu+1}$ if $M_\nu = 0$; finally for $\nu > \mu$, $c_\nu = M_\nu$ if $M_\nu \neq 0$ and $c_\nu = -\text{sg}\, c_{\nu-1}$ if $M_\nu = 0$. Then the number of sign changes of $g(x)$ in $0 < x < a$ is not less than even the number of sign changes in the sequence $c_0, c_1, \cdots, c_n$. See Pólya-Szegö, loc. cit. footnote 8, vol. 2, p. 50.

[15] I don't know to whom this identity is due. I have seen it stated and proved in the paper of E. Feldheim, *Contributions à la théorie des polynômes de Jacobi* (Hungarian with French summary), Matematikai és Physikai Lapok vol. 48 (1941) pp. 453–504.

[16] See, for example, G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, p. 96.

This together with the quoted theorem of Fejér shows that the number of positive sign changing places of $f(x)e^{-x}$ (or of $f(x)$) is not less than the number of sign changes in the sequence (10) or, what amounts to the same, in the sequence (6).

Of course a similar reasoning yields similar rules for other polynomial expansions but the rule given for Laguerre expansion seems to be particularly simple and to a certain extent dual to the rule of Laguerre (3). The question of constructing such simple rules leads to interesting questions in the theory of orthogonal polynomials; to these I shall return elsewhere.

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