A SIMPLE SOLUTION OF THE DIOPHANTINE EQUATION $x^3+y^3=z^2+t^2$

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1. Introduction. The problem of finding integral solutions for the equation $x^3+y^3=z^2+t^2$ was evidently proposed by E. Miot [1].[1] A. Cunningham [2] noted the obvious solutions $x=a^2$, $y=b^2$, $z=a^3$, $t=b^3$. Additional solutions, some of which were obtained by setting $x+y=l^2$, or $x+y=m^2+n^2$, and $x^2-xy+y^2=p^2+q^2$, were also given by Cunningham.

The equation has the obvious solutions $x=-y$, $z=t=0$. These are considered of little interest and are not discussed here. However, should it be desired to include these solutions, they may be given by $x=n$, $y=-n$, $z=t=0$, These same remarks apply to the general solution of $x^3+y^3=z^2$ given here as a direct consequence of the solution of $x^3+y^3=z^2+t^2$.

2. Solution. Let the equation first be written in the form

$$(x+y)[(2x-y)^2+3y^2]=4(z^2+t^2).$$

Hence, $(x+y)(2x-y+i\,3^{1/2}y)(2x-y-i\,3^{1/2}y)=4(z+it)(z-it)$.

Consider the system of equations,

$$n(x+y)=4s, \tag{1}$$

$$s(2x-y+i3^{1/2}y)=(p+mi+q3^{1/2}+ir3^{1/2})(z+it), \tag{2}$$

$$(p+mi+q3^{1/2}+ir3^{1/2})(2x-y-i3^{1/2}y)=n(z-it). \tag{3}$$

This system of equations will be satisfied in rational numbers if the following system, obtained from the system (1), (2), (3) by equating like components, is so satisfied. Hence, consider the system of equations,

$$nx+ny=4s, \tag{4}$$

$$2sx-sy-pz+mt=0, \tag{5}$$

$$sy-rz-qt=0, \tag{6}$$

$$mz+pt=0, \tag{7}$$

$$qz-rt=0, \tag{8}$$

$$2px+(3r-p)y-nz=0, \tag{9}$$

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[1] Numbers in brackets refer to bibliography given at end of paper.

$$2mx - (3q + m)y + nt = 0, \tag{10}$$

$$2qx + (m - q)y = 0, \tag{11}$$

$$2rx - (p + r)y = 0. \tag{12}$$

Now, consider the equations (5), (6), (7). All solutions of this system of equations are given by:

$$x = \mu_1 \begin{vmatrix} -s & -p & m \\ s & -r & -q \\ 0 & m & p \end{vmatrix} = \mu_1(prs + m^2s - mqs + p^2s), \tag{13}$$

$$y = \mu_1 \begin{vmatrix} 2s & m & -p \\ 0 & -q & -r \\ 0 & p & m \end{vmatrix} = 2\mu_1(prs - mqs), \tag{14}$$

$$z = \mu_1 \begin{vmatrix} 2s & -s & m \\ 0 & s & -q \\ 0 & 0 & p \end{vmatrix} = 2\mu_1(ps^2), \tag{15}$$

$$t = \mu_1 \begin{vmatrix} 2s & -p & -s \\ 0 & -r & s \\ 0 & m & 0 \end{vmatrix} = - 2\mu_1(ms^2). \tag{16}$$

In order that (7) and (8) may have nontrivial solutions for z and t it is clear that

$$m = - pq/r. \tag{17}$$

Hence

$$x = \mu \frac{(p + r)(r^2 + q^2)}{r^2}, \tag{18}$$

$$y = 2\mu \frac{(r^2 + q^2)}{r}, \tag{19}$$

$$z = 2\mu s, \tag{20}$$

$$t = 2\mu\, qs/r. \tag{21}$$

For (9) to be satisfied by the values of x, y, z, given by (18), (19), and (20), direct substitution shows that,

$$n = \frac{(r^2 + q^2)(p^2 + 3r^2)}{sr^2}. \tag{22}$$

With the condition (22), it may now be verified that the values of x, y, z, t given by (18), (19), (20) and (21) satisfy equations (5) through (12).

If equation (4) is to be satisfied by (18) and (19), it is found that

$$\mu = \frac{4s^2r^4}{(p+3r)(p^2+3r^2)(r^2+q^2)^2}.$$

Hence,

$$x = \frac{4s^2r^2(p+r)}{(p+3r)(p^2+3r^2)(r^2+q^2)}, \tag{23}$$

$$y = \frac{8s^2r^3}{(p+3r)(p^2+3r^2)(r^2+q^2)}, \tag{24}$$

$$z = \frac{8s^3r^4}{(p+3r)(p^2+3r^2)(r^2+q^2)^2}, \tag{25}$$

$$t = \frac{8s^3r^3q}{(p+3r)(p^2+3r^2)(r^2+q^2)^2}. \tag{26}$$

It is at once clear that s is completely at our disposal. Therefore, after choosing $s=(p+3r)(p^2+3r^2)(r^2+q^2)$, making the obvious simplifications and introducing the customary λ, we have

$$x = \lambda^2(p+r)(p+3r)(p^2+3r^2)(r^2+q^2), \tag{27}$$

$$y = 2\lambda^2 r(p+3r)(p^2+3r^2)(r^2+q^2), \tag{28}$$

$$z = \lambda^3 r(p+3r)^2(p^2+3r^2)^2(r^2+q^2), \tag{29}$$

$$t = \lambda^3 q(p+3r)^2(p^2+3r^2)^2(r^2+q^2). \tag{30}$$

3. **Completeness.** Let A, B, C, D be any values which satisfy the original equation; that is, $A^3+B^3=C^2+D^2$, and set

$$A = \lambda^2(p+r)(p+3r)(p^2+3r^2)(r^2+q^2), \tag{31}$$

$$B = 2\lambda^2 r(p+3r)(p^2+3r^2)(r^2+q^2), \tag{32}$$

$$C = \lambda^3 r(p+3r)^2(p^2+3r^2)^2(r^2+q^2), \tag{33}$$

$$D = \lambda^3 q(p+3r)^2(p^2+3r^2)^2(r^2+q^2). \tag{34}$$

It will now be shown that there exist values for p, q, r, and λ that will produce the given solution. In particular, if A, B, C, and D be integral, p, q, r are integral and λ rational. To this end consider the ratio,

$$C/D = r/q,$$

and choose $r=\lambda_1 C$, $q=\lambda_1 D$.

It now follows from the ratio A/B that

$$p = \frac{\lambda_1 C(2A - B)}{B}.$$

Let $\lambda_1 = B$. Hence,

$$r = BC, \tag{35}$$

$$q = BD, \tag{36}$$

$$p = C(2A - B). \tag{37}$$

Substituting (35), (36), and (37) in (31), (32), (33), and (34) we have

$$\begin{aligned}
A &= 16\lambda^2 AB^2C^4(C^2 + D^2)^2,\\
B &= 16\lambda^2 B^3C^4(C^2 + D^2)^2,\\
C &= 64\lambda^3 B^3C^7(C^2 + D^2)^3,\\
D &= 64\lambda^3 B^3C^6D(C^2 + D^2)^3.
\end{aligned}$$

Choosing $\lambda = 1/4BC^2(C^2+D^2)$, $B\neq 0$, $C\neq 0$, completes the proof. Moreover, it is clear that B and C may be chosen different from zero; for, if there be given solutions involving zeros, it is manifestly possible to set B and C equal to the nonzero members of these solutions, while A or D, or both A and D, may be chosen as zero. Hence, excluding the trivial solutions noted in the introduction, all solutions of the equation $x^3+y^3=z^2+t^2$ are given by (27), (28), (29), and (30).

It is worthwhile to note that the general solution of the equation $x^3+y^3=z^2$ now follows immediately. In (27), (28), (29) and (30) set $q=0$, and suppress the powers of r. Then,

$$\begin{aligned}
x &= \lambda^2(p + r)(p + 3r)(p^2 + 3r^2),\\
y &= 2\lambda^2 r(p + 3r)(p^2 + 3r^2),\\
z &= \lambda^3(p + 3r)^2(p^2 + 3r^2)^2.
\end{aligned}$$

The generality is at once clear, for (27), (28), (29), and (30) must give, in particular, all solutions of the form $x^3+y^3=z^2+0^2$. For a history of this case see [3, pp. 578–581].

The well known solution for the equation $y^3=z^2+t^2$ also follows as a special case. In (27), (28), (29) and (30) set $p=-r$, and make the indicated reductions. Then,

$$y = \lambda^2(r^2 + q^2), \qquad z = \lambda^3 r(r^2 + q^2), \qquad t = \lambda^3 q(r^2 + q^2).$$

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SQUARE ROOTS IN LOCALLY EUCLIDEAN GROUPS

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A possible attack on the fifth problem of Hilbert is to demonstrate the existence of one-parameter subgroups in any locally Euclidean group.[1] It is known that, provided there are no "small" subgroups, some one-parameter subgroups exist. One would like to prove, however, that in a suitable neighborhood of the identity, every element is on one and only one one-parameter subgroup. If this is true, it is possible to extract square roots (that is, solve $x^2=a$ for given a) uniquely in this neighborhood, and the sequence of successive square roots a, $a^{1/2}$, $(a^{1/2})^{1/2}$, $\cdots$ converges to the identity. Conversely, it is easily seen that, if unique square roots exist, and if the sequence of square roots converge to the identity, then the one-parameter subgroups can be found. In this paper we give a new proof[2] that square roots exist in a suitable neighborhood of the identity and show, in addition, that either they are unique or small subgroups exist.

Throughout this paper we shall deal only with locally Euclidean topological groups of dimension n; consequently we may speak of an n-cell neighborhood of a point p, meaning a homeomorphic image of a Euclidean n-simplex containing the point p in its interior.

The proofs of Theorems 1 and 2 use the group property so sparingly that they can easily be restated as theorems on the involutions of a manifold.

THEOREM 1. *In a locally Euclidean group there exists a neighborhood*

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[1] The arguments outlined here are those of B. von Kerekjarto in his paper *Geometrische Theorie der zweigliedrigen kontinuierlichen Gruppen*, Abh. Math. Sem. Hamburgischen Univ. vol. 8 (1930) pp. 107–114.

[2] Cf. B. de Kerekjarto, *Sur l'existence de racines carrées dans les groupes continus*, C. R. Acad. Sci. Paris vol. 193 (1931) pp. 1384–1385.