completely additive functions on σ-fields and of the theory of Carathéodory.

The first chapter deals with the basic properties of additive and totally additive set functions, zero sets and complete fields, and the decomposition into regular and singular parts. Chapter II is devoted to the Carathéodory theory of measure. Special attention is paid to regular measures, and to n-dimensional Lebesgue measure. In the third chapter the author discusses the properties of measurable functions and sequences of such functions. The theory of integration is developed in Chapter IV by characterizing the indefinite integral as a new measure satisfying certain inequalities. A discussion of the approximation of integrals by sums, some mean value theorems, and convergence theorems, is followed by a section on product measures and the Fubini theorem. The last chapter deals with the Vitali covering theorem, the differentiation of measures and interval functions, and some applications to density and approximate continuity.

The book is carefully written and systematic. The proofs are given in great detail, a fact which may help many who wish to become acquainted with the fundamentals of measure theory.

HERBERT FEDERER

Sur les groupes classiques. By Jean Dieudonné. (Actualités scientifiques et industrielles, no. 1040; Publications de l'Institut de Mathématiques de l'Université de Strasbourg. VI.) Paris, Hermann, 1948. 82 pp.

The main purpose of this little book is to obtain the structural properties of the classical groups which can at present be obtained by purely algebraic methods. By skillful organization, complete mastery of his subject, and constant adherence to the "conceptual" point of view so fruitfully introduced into linear algebra in modern times, the author achieves this purpose with simplicity, efficiency, and elegance. The results are, with some exceptions, either old ones (to be found in the pioneering works of L. E. Dickson), or extensions of old ones to more general situations. But the long complicated matrix computations of the older literature, in which the ideas are frequently buried beyond recall, are here almost entirely replaced by conceptual arguments expressed in geometric language which brings out for inspection the intuitive geometric motivation in the proofs.

The classical groups are the full linear groups $GL_n(K)$, the symplectic groups $Sp_n(K)$, the orthogonal groups $O_n(K, f)$, and the unitary groups $U_n(K, f)$. The full linear groups over an arbitrary skew field K have been previously treated by the author (Bull. Soc. Math. France vol. 71 (1943) pp. 27–45), and therefore are not considered in the present book. The symplectic and orthogonal groups are defined over a completely arbitrary field K, and the unitary groups either over a separable extension K of degree 2 over an arbitrary field or over a non-commutative skew field K possessing an involutorial anti-automorphism. Most of this great generality comes gratis with the proofs; but a large proportion of pages is required to clear up certain "marginal" cases (characteristic 2, fields with 3 elements, and so on).

The symplectic groups prove to be the easiest. A bilinear form f on a vector space E of dimension n over K is *alternating* if $f(x, x)=0$ identically; the rank of such an f is always even. Given such a form of rank n (so that $n=2m$), a linear transformation u of E onto E is *symplectic* if $f(ux, uy)=f(x, y)$ identically; the group of all symplectic transformations of E is $Sp_n(K)$. It is always possible to reduce to the case $f(x, y)=\sum_{i=1}^{m}(\xi_i\eta_{m+i}-\xi_{m+i}\eta_i)$ by choosing a suitable base of E; therefore up to isomorphism $Sp_n(K)$ does not depend on the particular choice of the alternating form f. The structure of $Sp_n(K)$ is determined by proving: (a) the center of $Sp_n(K)$ consists of the identity transformation I and $-I$ (easy); (b) except when K has 2 elements and $n=2$ or 4, or when K has 3 elements and $n=2$, the factor group of $Sp_n(K)$ by its center is simple. The proof of (b) largely depends on the one hand on the fact that $Sp_2(K)$ is identical with the unimodular group $SL_2(K)$ which, by the author's above-mentioned paper on $GL_2(K)$, is known to be simple modulo its center except when the number of elements in K is 2 or 3, and on the other hand on the author's theorem that every symplectic transformation is a product of *symplectic transvections*, that is, mappings of the form $x\rightarrow x+\lambda f(x, a)$ (where $\lambda\in K$, $a\in E$, $f(a, a)=0$).

The structure of the orthogonal groups proves to be more complicated. In the case of field characteristic other than 2, a symmetric bilinear form f on E uniquely determines a quadratic form g, and conversely. A transformation u of E onto E is said to be *orthogonal* if $g(ux)=g(x)$ (or equivalently $f(ux, uy)=f(x, y)$). The group of all such orthogonal transformations is $O_n(K, f)$. In general, two quadratic forms are not equivalent, so that it is necessary to specify the particular f used. A vector subspace V of E is *totally isotropic* if $f(x, y)=0$ for all x and y in V. The maximum dimension ν of a totally isotropic subspace is the *index* of f. (This index, introduced by E. Witt (J. Reine Angew. Math. vol. 176 (1937) pp. 31–44), in an ordered field becomes the smaller of the two numbers s and $n-s$, where s is the index of inertia of g.) In the case of $Sp_n(K)$ the integer analogous to ν is $n/2$; much of the complication surrounding $O_n(K,f)$ seems to arise from the fact that ν can assume any value subject to $0 \leqq \nu \leqq n/2$. When $n \geqq 5$ and $\nu \geqq 1$ the structure can be described as follows: (a) the group $O_n^+(K, f)$ of all orthogonal transformations with determinant unity is a normal subgroup of $O_n(K,f)$ of index 2; (b) the commutator group $\Omega_n(K,f)$ of $O_n(K,f)$ is a normal subgroup of $O_n^+(K,f)$ with factor group isomorphic with a subgroup of K^*/Q, where K^* is the multiplicative group of nonzero elements of K and Q is the group of all squares of elements of K^*; (c) the factor group $\Omega_n(K,f)/Z_n \cap \Omega_n(K,f)$, where Z_n is the center of $O_n(K,f)$, is simple; (d) $Z_n = \{I, -I\}$. The proof depends on a theorem on quadratic forms due to Witt (loc. cit.), on certain detailed information concerning $O_n(K, f)$ for $2 \leqq n \leqq 6$ quoted from B. L. van der Waerden's *Gruppen von linearen transformationen* (Berlin, 1935), and on a sequence of interesting propositions concerning orthogonal equivalence of subspaces of E and sets of generators for $O_n(K, f)$, $O_n^+(K, f)$, and $\Omega_n(K, f)$. When $\nu = 0$, however, the structure of $O_n(K, f)$ evidently depends in a deep way on the nature of the particular field K, and the author confines himself to examples illustrating this fact. When K is the field of real numbers and f is definite (so that $\nu = 0$) then (for $n > 2$) $\Omega_n(K, f) = O_n^+(K, f)$ and $O_n^+(K, f)/Z_n$ is known to be simple; but when K is the field of formal power series $\sum_{k=h}^{\infty} \lambda_k t^k$ with real coefficients then the author shows that $O_n(K,f)$ contains a sequence $O_n(K,f) = G_0 \supset G_1 \supset \cdots$ of normal subgroups such that $\bigcap_{i \geqq 0} G_i = \{I\}$ and each factor group G_{i-1}/G_i is abelian.

The case of field characteristic 2 presents special problems. A quadratic form g is defined as a mapping g of E into K such that $g(\lambda x + \mu y) = \lambda^2 g(x) + \mu^2 g(y) + \lambda\mu f(x, y)$ identically, where f is a bilinear form on E. f is then alternating and therefore is of even rank $2p \leqq n$; the set E^* of all vectors x such that $f(x, y) = 0$ for all y in E is a vector space of dimension $n - 2p$; the set E_0 of all vectors x in E^* such that $g(x) = 0$ is a vector space of dimension say $q \leqq n - 2p$. g is called *regular* if $q = 0$; only regular quadratic forms are considered. The index ν is now defined as the maximum dimension of a totally singular subspace of E, a vector subspace V being totally singular if $g(x) = 0$ for all x in V. $O_n(K, g)$ is now defined as the group of all linear transformations u of E onto E such that $g(ux) = g(x)$ identically. When $n - 2p = 0$ (*defect* 0), $n \geqq 6$, and $\nu \geqq 1$, and when $n - 2p > 0$ (*defect* > 0), $2p \geqq 2$, $\nu \geqq 1$ (with the possible exception of the case $2p = 4$, $\nu = 2$), the author proves that the commutator group $\Omega_n(K, g)$ is simple. Moreover, he shows that $O_n(K, g)/\Omega_n(K, g)$ is, in the case of defect 0, isomorphic with a subgroup of K^*/Q, as in the case of characteristic not 2, or with the product of such a subgroup and a cyclic group of order 2. Examples are given which show that when $\nu=0$ the structure need not be so simple.

Turning to the unitary groups the author considers two very general types. To define the first type, consider a separable extension K of degree 2 over an arbitrary field K_0. By means of the unique automorphism $\xi\rightarrow\bar{\xi}$ of K over K_0, the concept of *hermitian symmetric form* is introduced: A mapping f of $E\times E$ into K is an hermitian symmetric form if $f(x+x', y)=f(x, y)+f(x', y)$, $f(x, y+y')=f(x, y)+f(x, y')$, $f(\lambda x, \mu y)=\bar{\lambda}\mu f(x, y)$, $f(y, x)=\overline{f(x, y)}$ identically. Given a symmetric hermitian form f of rank n, a linear transformation u of E onto E is *unitary* if $f(ux, uy)=f(x, y)$ identically, and the group of all such unitary transformations is $U_n(K, f)$. For the second type of unitary group the starting point is a *reflexive* non-commutative skew field K, that is, a skew field K distinct from its center K_0 for which there exists an involutorial antiautomorphism $\xi\rightarrow\bar{\xi}$ relative to K_0 such that $\xi+\bar{\xi}$ and $\xi\bar{\xi}$ belong to K_0 for every ξ in K. A reflexive skew field is always of rank 4 over its center (for characteristic not 2 is a generalized quaternion skew field). If E is a right vector space over K then a mapping f of $E\times E$ into K is an *hermitian symmetric form* on E if $f(x+x', y)=f(x, y)+f(x', y)$, $f(x, y+y')=f(x, y)+f(x, y')$, $f(x\lambda, y\mu)=\bar{\lambda}f(x, y)\mu$, $f(y, x)=\overline{f(x, y)}$ identically, and $f(x, x)$ always belongs to the center K_0. The definition of $U_n(K, f)$ then proceeds as in the commutative case. For both types the theory develops much as for $O_n(K, f)$, except that characteristic 2 does not cause so much difficulty as before. The index ν is defined in the expected way. In the commutative case, when $n\geqq 2$ and $\nu\geqq 1$, the structure can be described as follows: (a) the set $U_n^+(K, f)$ of unitary transformations with determinant unity is a normal subgroup of $U_n(K, f)$ with factor group isomorphic with the multiplicative group of all elements $\lambda\in K$ such that $\lambda\bar{\lambda}=1$; (b) except when K_0 has 3 elements and $n=2$ and when K_0 has 2 elements and $n=2$ or 3, the factor group $U_n^+(K, f)/Z_n$, where Z_n is the center of $U_n^+(K, f)$, is simple; (c) Z_n consists of those mappings $x\rightarrow\lambda x$ for which $\lambda^n=1$ and $\lambda\bar{\lambda}=1$. In the non-commutative case for $n\geqq 2$ and $\nu\geqq 1$, the factor group of $U_n(K, f)$ by its center Z_n is simple, and $Z_n=\{I, -I\}$. In both cases the structure is irregular for $\nu=0$.

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