SUMMABILITY WITH A GOVERNOR OF INTEGRAL ORDER[1]

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1. Introduction. In this paper the method of summability with a governor, introduced by G. Piranian [5],[2] is generalized, and the effectiveness of the new methods is compared with that of the Cesàro methods of integral order.

2. Preliminary theorems. We shall consider throughout this paper an arbitrary formal series

$$\text{(A)} \qquad \sum_{j=0}^{\infty} a_j.$$

Using the notation $s_n = \sum_{j=0}^{n} a_j$, we consider the standard Toeplitz transformation

$$\text{(1)} \qquad \sigma_n = \sum_{j=0}^{\infty} b_{nj} s_j.$$

The following theorem is well known.

THEOREM 1 (SILVERMAN-TOEPLITZ [8, 9]). *A necessary and sufficient condition that the transformation* (1) *be regular is that the following hold*:

$$(\alpha) \qquad \sum_{j=0}^{\infty} | b_{nj} | < M, \quad M \textit{ independent of } n,$$

$$(\beta) \qquad \lim_{n\to\infty} b_{nj} = 0 \quad \textit{for all } j,$$

$$(\gamma) \qquad \lim_{n\to\infty} \sum_{j=0}^{\infty} b_{nj} = 1.$$

Consider now an arbitrary sequence $\{\lambda_n\}$ of nonnegative terms, not all zero, and set $\Lambda_n = \sum_{j=0}^{n} \lambda_j$. Then $\Lambda_n > 0$ for all sufficiently large n. If, for these n, we let $b_{nj} = \lambda_{n-j}/\Lambda_n$ for $j \leqq n$ and $b_{nj} = 0$ for $j > n$, then (1) becomes

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

$$(2) \qquad \sigma_n = \frac{1}{\Lambda_n} \sum_{j=0}^{n} s_j \lambda_{n-j}.$$

(For the finite number of cases in which $\Lambda_n = 0$ we define σ_n to be zero.)

If $\lim_{n\to\infty} \sigma_n$ exists we say that (A) is Nörlund summable by means of the sequence $\{\lambda_n\}$, or summable (N, λ) [4]. It is easily seen from Theorem 1 that (N, λ) is regular if and only if $\lambda_n = o(\Lambda_n)$. We shall consider only regular Nörlund methods.

It is known that any two regular Nörlund methods, (N, λ) and (N, ω), are consistent. If, in particular, every series summable (N, ω) is summable (N, λ) we shall write $(N, \omega) \subset (N, \lambda)$.

THEOREM 2 (M. RIESZ [7]). *A necessary and sufficient condition that $(N, \omega) \subset (N, \lambda)$, where $\omega_0 > 0$, is that*

$$\sum_{j=0}^{n} \Omega_{n-j} | q_j | = O(\Lambda_n),$$

and $q_n = o(\Lambda_n)$, where $\Omega_n = \sum_{j=0}^{n} \omega_j$ and $\{q_j\}$ is defined by the formal relation

$$\sum_{j=0}^{\infty} q_j x^j = \sum_{j=0}^{\infty} \lambda_j x^j \Big/ \sum_{j=0}^{\infty} w_j x^j.$$

If we choose $\omega_n = C_{n+k-1,k-1}$ (k a positive integer) the method (N, ω) becomes the classical Cesàro mean of order k, or (C, k) [1].

Defining $\Delta\lambda_n = \lambda_n - \lambda_{n-1}$, and, inductively, $\Delta^k\lambda_n = \Delta^{k-1}\lambda_n - \Delta^{k-1}\lambda_{n-1}$ with $\lambda_n = 0$ for $n < 0$, we obtain the following results:

COROLLARY 1. *A necessary and sufficient condition that $(C, k) \subset (N, \lambda)$ is that*

$$(3) \qquad \sum_{j=0}^{n} C_{n-j+k,k} | \Delta^k \lambda_j | = O(\Lambda_n).$$

To see this we observe that

$$\sum_{j=0}^{\infty} w_j x^j = \sum_{j=0}^{\infty} C_{j+k-1,k-1} x^j = (1 - x)^{-k},$$

so that

$$\sum_{j=0}^{\infty} q_j x^j = (1 - x)^k \sum_{j=0}^{\infty} \lambda_j x^j = \sum_{j=0}^{\infty} (\Delta^k \lambda_j) x^j.$$

Hence $q_j = \Delta^k \lambda_j$. Further, $\Omega_n = \sum_{j=0}^{n} C_{j+k-1,k-1} = C_{n+k,k}$. Finally, since

$\lambda_n = o(\Lambda_n)$, it is clear that $\Delta^k\lambda_n = o(\Lambda_n)$, and Theorem 2 yields the result.

COROLLARY 2. *A sufficient condition that $(C, k) \subset (N, \lambda)$ is that $\Delta^k\lambda_n \geqq 0$ for all n.*

Here we have

$$\begin{aligned}\sum_{j=0}^{n} C_{n-j+k,k} \left| \Delta^k\lambda_j \right| &= \sum_{j=0}^{n} C_{n-j+k,k}\{\Delta^{k-1}\lambda_j - \Delta^{k-1}\lambda_{j-1}\} \\ &= \sum_{j=0}^{n-1} \{C_{n-j+k,k} - C_{n-j-1+k,k}\}\Delta^{k-1}\lambda_j + \Delta^{k-1}\lambda_n \\ &= \sum_{j=0}^{n} C_{n-j+k-1,k-1}\Delta^{k-1}\lambda_j.\end{aligned}$$

Repeating the argument k times we find

$$\sum_{j=0}^{n} C_{n-j+k,k} \left| \Delta^k\lambda_j \right| = \sum_{j=0}^{n} \lambda_j = \Lambda_n,$$

so that (3) is satisfied.

One more useful corollary is now obtainable.

COROLLARY 3.[3] *If the sequence $\{\lambda_n\}$ is such that the sequence $\{\Delta^k\lambda_n\}$ $(n = 0, 1, \cdots)$ has at least one but only a finite number of negative elements, a necessary and sufficient condition that $(C, k) \subset (N, \lambda)$ is that $n^k = O(\Lambda_n)$.*

Let $\Delta^k\lambda_n \geqq 0$ for all $n > N_0$. We write

$$\begin{aligned}\sum_{j=0}^{n} C_{n-j+k,k} \left| \Delta^k\lambda_j \right| &= \sum_{j=0}^{n} C_{n-j+k,k}\Delta^k\lambda_j \\ &\quad + \sum_{j=0}^{N_0} C_{n-j+k,k}\{ \left| \Delta^k\lambda_j \right| - \Delta^k\lambda_j\} \\ &= \Lambda_n + \sum_{j=0}^{N_0} C_{n-j+k,k}\{ \left| \Delta^k\lambda_j \right| - \Delta^k\lambda_j\}.\end{aligned}$$

Since $C_{n,k} \sim n^k/k!$ the result is obvious.

3. **The method** (G, k). We shall henceforth consider the series (A) under the additional stipulation that not all $a_n = 0$. We define

[3] A weaker result was originally stated. The author is indebted to the referee for the improved formulation.

$p_n(0)=|a_n|$, and, by induction, $p_n(k)=\sum_{j=0}^{n} p_j(k-1)$, and set

$$G_n(k) = \frac{1}{p_n(k+1)} \sum_{j=0}^{n} s_j p_{n-j}(k) \qquad (k \geqq 1). \tag{4}$$

(For the finite number of cases for which $p_n(k+1)$ may be zero we define $G_n(k)$ to be zero.)

DEFINITION. If $\lim_{n\to\infty} G_n(k)=\sigma$ exists and if $\lim_{n\to\infty} p_n(k)/p_n(k+1)=0$, the series (A) is summable to σ by means of a governor of order k, or (G, k) summable to σ.[4]

The method (G, k) reduces to the method (G) of Piranian when $k=1$. It is easily shown to be a regular Nörlund method. A stronger result is the following:

THEOREM 3. *For every positive integer* k, $(G, k)\subset(G, k+1)$.

Let (A) be (G, k) summable to σ. Since

$$\begin{aligned} G_n(k+1) &= \frac{1}{p_n(k+2)} \sum_{j=0}^{n} s_j p_{n-j}(k+1) \\ &= \frac{1}{p_n(k+2)} \sum_{j=0}^{n} s_j \sum_{i=0}^{n-j} p_i(k) \\ &= \frac{1}{p_n(k+2)} \sum_{m=0}^{n} \sum_{r=0}^{m} s_r p_{m-r}(k) \\ &= \sum_{m=0}^{n} G_m(k) p_m(k+1) \Big/ \sum_{m=0}^{n} p_m(k+1), \end{aligned}$$

the sequence $\{G_n(k+1)\}$ is obtained from the sequence $\{G_n(k)\}$ by a Riesz transformation (R, q) with $q_n=p_n(k+1)$ [2, 6]. Further, $\lim_{n\to\infty} p_n(k+1)=+\infty$ so that the transformation is regular and $\lim_{n\to\infty} G_n(k+1)=\sigma$.

That $\lim_{n\to\infty} p_n(k+1)/p_n(k+2)=0$ follows from a result of Piranian [5, Lemma]. Thus (A) is $(G, k+1)$ summable.

All of the results known for summability (G) are readily extended to the case (G, k), although the details are frequently tedious. Since the topic of greatest interest here is the relation between Cesàro summability and the new method we shall not investigate these extensions. The next theorem will, however, prove useful in our work. When two or more series are under consideration we shall use

[4] For the sake of completeness we may agree that a series with all zero terms is (G, k) summable to the value zero for all positive integers k. This will be entirely consistent with all further work.

the notations $p_n(k;\ a)$, $s_n(a)$, $G_n(k;\ a)$, and so on, to refer to the series (A).

THEOREM 4. *Let* (A) *be such that* $\lim_{n\to\infty} G_n(k;\ a)=\sigma$ *and*

$$\lim_{n\to\infty} \sum_{j=0}^{n} C_{n-j+k-1,k-1}s_j(a)/p_n(k+1;\ a) = 0. \tag{5}$$

Let

$$\sum_{j=0} b_j \tag{B}$$

be such that $b_j=a_j$ *for* $j\neq i$ (i *fixed*) *and* $b_i=a_i+h$. *Then, in order that* $\lim_{n\to\infty}G_n(k;\ b)=\sigma+h$, *it is necessary and sufficient that the condition* $\lim_{n\to\infty}p_n(k;\ a)/p_n(k+1;\ a)=0$ *hold.*

We first observe that

$$\begin{aligned} p_n(r) &= \sum_{j=0}^{n} C_{n-j+r-2,r-2}p_j(1) \\ &= \sum_{j=0}^{n} C_{n-j+r-1,r-1}\,|\,a_j\,|, \end{aligned} \tag{6}$$

as may easily be verified by induction. Thus

$$s_n(b) = s_n(a); \qquad p_n(r;\ b) = p_n(r;\ a) \qquad (n < i;\ r \geqq 0);$$
$$s_n(b) = s_n(a) + h;$$
$$p_n(r;\ b) = p_n(r;\ a) + \{\,|\,a_i + h\,| - |\,a_i\,|\,\}C_{n-i+r-1,r-1} \qquad (n \geqq i;\ r \geqq 1).$$

Hence, for $n\geqq i$,

$$\frac{p_n(1;\ b)}{p_n(1;\ a)} = 1 + \frac{\{\,|\,a_i+h\,| - |\,a_i\,|\,\}}{p_n(1;\ a)},$$

while for $n\geqq i$, $r\geqq 2$,

$$\frac{p_n(r;\ b)}{p_n(r;\ a)} = 1 + \{\,|\,a_i+h\,| - |\,a_i\,|\,\}\frac{C_{n-i+r-1,r-1}}{C_{n+r-1,r-1}} \cdot \left\{\frac{C_{n+r-1,r-1}}{\sum_{j=0}^{n} C_{n-j+r-2,r-2}p_j(1;\ a)}\right\}.$$

Since only the case of divergent series is of interest we may assume that $\lim_{n\to\infty} p_n(1;\ a)=+\infty$, so that

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} C_{n-j+r-2,r-2}p_j(1;a)}{C_{n+r-1,r-1}} = +\infty. \tag{7}$$

Therefore, $p_n(r;b)=\{1+\epsilon_n(r)\}p_n(r;a)$, where $\lim_{n\to\infty}\epsilon_n(r)=0$, $r\geqq 1$. Now, for $n>i$,

$$\begin{aligned}\sum_{j=0}^{n} s_j(b)p_{n-j}(k;b)& \\ &= \sum_{j=0}^{n-i} s_j(a)\{p_{n-j}(k;a)+[|a_i+h|-|a_i|]C_{n-i-j+k-1,k-1}\} \\ &\quad + \sum_{j=n-i+1}^{n} s_j(a)p_{n-j}(k;a) + h\sum_{j=i}^{n} p_{n-j}(k;b).\end{aligned}$$

Thus

$$\begin{aligned}G_n(k;b) &= \frac{G_n(k;a)}{1+\epsilon_n(k+1)} + h\left\{1-\frac{1}{p_n(k+1;b)}\sum_{j=n-i+1}^{n} p_j(k;b)\right\} \\ &\quad + \frac{\{|a_i+h|-|a_i|\}}{1+\epsilon_n(k+1)}\,\frac{p_{n-i}(k+1;a)}{p_n(k+1;a)}\,\frac{\sum_{j=0}^{n-i} C_{n-i-j+k-1,k-1}s_j(a)}{p_{n-i}(k+1;a)}.\end{aligned}$$

Since $0\leqq p_{n-i}(k+1;a)/p_n(k+1;a)\leqq 1$ the third term approaches zero, by (5). The result now follows readily.

4. **Relation between (G, k) and (C, k) summability.**

THEOREM 5. *For every positive integer k, $(C, k)\subset(G, k)$.*

Let us assume that (A) is summable (C, k) to σ. That is, we have $\lim_{n\to\infty} T_n(k)=\sigma$, where $T_n(k)=S_n(k)/C_{n+k,k}$ with $S_n(k) = \sum_{j=0}^{n} C_{n-j+k-1,k-1}\, s_j$. From (6) we have

$$\begin{aligned}\sum_{j=0}^{n} s_jp_{n-j}(k) &= \sum_{j=0}^{n} s_j\sum_{r=0}^{n-j} C_{r+k-1,k-1}|a_{n-j-r}| \\ &= \sum_{j=0}^{n} |a_{n-j}| \sum_{r=0}^{j} C_{j-r+k-1,k-1}s_r \\ &= \sum_{j=0}^{n} |a_{n-j}|\, C_{j+k,k}T_j(k).\end{aligned} \tag{8}$$

Thus

$$G_n(k) = \frac{\sum_{j=0}^{n} | a_{n-j} | C_{j+k,k} T_j(k)}{\sum_{j=0}^{n} | a_{n-j} | C_{j+k,k}}. \tag{9}$$

Now (9) represents a Toeplitz transformation applied to the convergent sequence $\{T_n(k)\}$ where, in the notation of Theorem 1,

$$b_{ni} = \frac{| a_{n-i} | C_{i+k,k}}{\sum_{j=0}^{n} | a_{n-j} | C_{j+k,k}} \qquad (i \leqq n),$$

$$b_{ni} = 0 \qquad (i > n).$$

That conditions (α) and (γ) of Theorem 1 are satisfied is evident. We need investigate only condition (β).

Let $a_m \neq 0$. Then, for $n \geqq m$,

$$\sum_{j=0}^{n} | a_{n-j} | C_{j+k,k} \geqq | a_m | C_{n-m+k,k}.$$

Hence

$$b_{ni} \leqq \frac{| a_{n-i} | C_{i+k,k}}{| a_m | C_{n-m+k,k}} \qquad (n \geqq \max (i, m)).$$

But, since (A) is summable (C, k), $a_n = o(n^k)$ [3, p. 484]. Therefore, $\lim_{n\to\infty} b_{ni} = 0$ for all i. Thus (9) is regular and $\lim_{n\to\infty} G_n(k) = \sigma$.

The condition $\lim_{n\to\infty} p_n(k)/p_n(k+1) = 0$ must still be established. We consider the series (B). Since (A) is (C, k) summable to σ, (B) is (C, k) summable to $\sigma + h$. Further,

$$\lim_{n\to\infty} \frac{1}{p_n(k+1; a)} \sum_{j=0}^{n} C_{n-j+k-1,k-1} s_j(a) = \lim_{n\to\infty} T_n(k; a) \left\{ \frac{C_{n+k,k}}{p_n(k+1; a)} \right\}$$
$$= 0,$$

since the bracketed expression approaches zero (see (7)). Therefore Theorem 4 applies and $\lim_{n\to\infty} p_n(k; a)/p_n(k+1; a) = 0$.

THEOREM 6. *For every integer $k > 1$ there exist series (G, k), but not $(G, k-1)$, summable.*[5]

We establish this result by example. Choosing the integer m such that $m > k+1$ we consider the series $\sum_{j=0}^{\infty} d_j$ for which

[5] That there are divergent series which are $(G, 1)$ summable is evident from Theorem 5.

$$S_{mn}(k-1) = C_{n+k-1,k-1} \qquad (n = 0, 1, 2, \cdots),$$
$$S_p(k-1) = 0 \qquad (p \neq mn).$$

Recalling that $S_n(1) = \sum_{j=0}^{n} s_j$, $S_n(r) = \sum_{j=0}^{n} S_j(r-1)$ for $r > 1$ [3, p. 466], we compute

$$S_{mn}(k-2) = C_{n+k-1,k-1} \qquad (n = 0, 1, 2, \cdots),$$
$$S_{mn+1}(k-2) = -C_{n+k-1,k-1},$$
$$S_p(k-2) = 0 \qquad (p \neq mn \text{ or } mn+1).$$

Repeating the argument we eventually find

$$s_{mn+j} = (-1)^j C_{k-1,j} C_{n+k-1,k-1} \quad (n = 0, 1, 2, \cdots; 0 \leqq j \leqq k-1),$$
$$s_p = 0 \qquad (p \neq mn + j),$$

with

$$d_{mn+j} = (-1)^j C_{k,j} C_{n+k-1,k-1} \qquad (n = 0, 1, 2, \cdots; 0 \leqq j \leqq k),$$
$$d_p = 0 \qquad (p \neq mn + j).$$

We notice in particular that $d_{mn-1} = 0$ $(n = 1, 2, 3, \cdots)$. Therefore, $\sum_{j=0}^{mn-1} |d_j| S_{mn-j-1}(k-1) = 0$. To see this we observe that $S_{mn-j-1}(k-1) \neq 0$ only when $mn - j - 1 = mp$, p an integer, and here $j = m(n-p) - 1$, so that $d_j = 0$. Therefore, by (8), $G_{mn-1}(k-1) = 0$ $(n = 1, 2, 3, \cdots)$. Hence, if $\lim_{n\to\infty} G_n(k-1) = \sigma$ exists, then $\sigma = 0$.

But, for r such that $mn \leqq r < m(n+1)$, $S_r(k) = C_{n+k,k}$, and thus

$$\frac{C_{n+k,k}}{C_{m(n+1)+k,k}} < \frac{S_r(k)}{C_{r+k,k}} = T_r(k) \leqq \frac{C_{n+k,k}}{C_{mn+k,k}}.$$

Therefore $\lim_{n\to\infty} T_n(k) = 1/m^k$, so that $\sum_{j=0}^{\infty} d_j$ is (C, k) summable. By Theorem 5, then, the series is also (G, k) summable and $\lim_{n\to\infty} G_n(k) = 1/m^k$. Finally, if $\sum_{j=0}^{\infty} d_j$ were also $(G, k-1)$ summable to σ, then σ would be $1/m^k$. This cannot be the case and our series therefore has the desired properties.

We have seen that the method (G, k) is at least as effective as the method (C, k). We shall now determine conditions under which (G, k) is stronger than the corresponding Cesàro mean.

THEOREM 7. *Let* (A) *be* (C, k) *summable. Then, in order that* (A) *be* (G, r) *summable*, $r < k$, *it suffices that* $\lim_{n\to\infty} p_n(r)/p_n(r+1) = 0$ *and that any one of the following conditions holds*:

$$\text{(I)} \qquad \sum_{j=0}^{n} C_{n-j+k,k} \left| \Delta^{k-r}(|a_j|) \right| = O(p_n(r+1));$$

$$\text{(II)} \qquad \Delta^{k-r}(|a_n|) \geqq 0 \quad \textit{for all } n;$$

(III) *There exists an* N_0 *such that* $\Delta^{k-r}(|a_n|) \geqq 0$ *for all* $n > N_0$ *and* $n^k = O(p_n(r+1))$.

To obtain these results from the corollaries of Theorem 2 we merely notice that

$$\Delta^k p_n(r) = \Delta^{k-1}\{p_n(r) - p_{n-1}(r)\} = \Delta^{k-1}p_n(r-1)$$
$$= \cdots = \Delta^{k-r+1}p_n(1) = \Delta^{k-r}(|a_n|).$$

It is interesting to observe that Theorem 5 could be obtained from this result if a direct proof that (C, r) summability implies $\lim_{n\to\infty} p_n(r)/p_n(r+1) = 0$ were available. The author has been unable to find such a proof.

We conclude by giving a new proof of a theorem due to Piranian [5, Theorem 8].

THEOREM 8 (PIRANIAN). *Let* $f(x)$ *be a polynomial with real coefficients. Then*

$$\text{(10)} \qquad \sum_{j=0}^{\infty} (-1)^j f(j)$$

is $(G, 1)$ *summable.*

Let $f(x) = a_0x^k + a_1x^{k-1} + \cdots + a_k$, where $a_0 > 0$. It is known that (10) is $(C, k+1)$ summable [3, p. 479], and that $\Delta^k f(n) = a_0 k!$ for $n > k$. Further, there exists an N_0 such that $f(n) > 0$ for all $n > N_0$. Hence, for $n > \max (k, N_0)$, $\Delta^k(|f(n)|) > 0$. Finally, we have $p_n(2) \sim a_0 n^{k+2}/(k+1)(k+2)$ and Theorem 7 (III) yields the result.

For $a_0 < 0$ we need only consider $g(x) = -f(x)$.

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ON THE REPRESENTATION OF A FUNCTION AS A HELLINGER INTEGRAL

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We derive in this note a necessary and sufficient condition that a nondecreasing, continuous function h of a single variable x be representable as a Hellinger integral of the form $\int_0^x (df)^2/dg$. This condition was first proved by Hellinger in his dissertation [1].[1] Other proofs have been given by Hahn [2] and Hobson [3], who transform to Lebesgue integrals and make use of Lebesgue theory. Hellinger's proof and the less complicated proof given here have a certain simplicity in that they avoid reliance on these notions and even remain entirely within the range of monotone functions.

We consider nondecreasing functions of a real variable x on the interval $0 \leqq x \leqq 1$ (henceforth denoted as [0, 1]). For such a function $f(x)$ and a closed interval Δ with end points x_1 and x_2 ($x_1 \leqq x_2$), we define a new function $f_\Delta(x)$ to denote the length of the interval on the f-axis determined by the interval on the x-axis common to Δ and $(0, x)$. More precisely, denoting

$$f(x \pm 0) = \lim_{h \to 0} f(x \pm |h|) \qquad \text{if } 0 < x < 1,$$

$$f(0-0) = f(0); \qquad f(1+0) = f(1),$$

we define

$$(1) \qquad f_\Delta(x) = \begin{cases} 0 & \text{if } 0 \leqq x < x_1, \\ f(x+0) - f(x_1 - 0) & \text{if } x_1 \leqq x \leqq x_2, \\ f(x_2 + 0) & \text{if } x_2 < x \leqq 1. \end{cases}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.