

PROOF OF A THEOREM OF SAKS AND SIERPINSKI

CASPER GOFFMAN

The object of this note is to give a direct proof of the following theorem of Saks and Sierpinski:

If $f(x)$ is an arbitrary one-valued real function defined on the closed interval $I = [0, 1]$, there is a $\phi(x)$ of Baire class 2 at most such that for every $\epsilon > 0$ the inequality $|f(x) - \phi(x)| < \epsilon$ holds on a set of exterior measure 1.

The proofs in the literature [1, 2]¹ depend on the corresponding theorem for measurable functions; namely, if $f(x)$ is measurable, there is a $\phi(x)$ of Baire class 2 at most such that $f(x) = \phi(x)$ almost everywhere.

We first prove a lemma which seems to be new.

LEMMA. *If $f(x)$ is defined on the closed interval $I = [0, 1]$, $\epsilon > 0$, $\phi(x)$ continuous, and $|f(x) - \phi(x)| < \epsilon$ on a set of exterior measure greater than $1 - \epsilon$, then for every $\eta > 0$ there is a continuous $\psi(x)$ such that $|\phi(x) - \psi(x)| < \epsilon$ on a set of measure greater than $1 - \epsilon$ and $|f(x) - \psi(x)| < \eta$ on a set of exterior measure greater than $1 - \eta$.*

PROOF. $f(x)$ is exteriorly approximately continuous almost everywhere in the sense that for almost every $\xi \in I$ the set of points x for which $f(\xi) - k < f(x) < f(\xi) + k$ has exterior metric density 1 at ξ , for every $k > 0$. There is a δ , with $0 < \delta < \eta$, such that $|f(x) - \phi(x)| < \epsilon - \delta$ on a set E of exterior measure greater than $1 - \epsilon$. Since $f(x)$ is exteriorly approximately continuous almost everywhere, every $\xi \in E - Z$, with Z of measure zero, is in a sequence, $\{I_{\xi_n}\}$, $i = 1, 2, \dots$, of closed intervals, whose lengths converge to zero, such that the set of points x for which $|f(\xi) - f(x)| < \delta$ has relative exterior measure exceeding $1 - \delta/2$ in each I_{ξ_n} . Moreover, since $\phi(x)$ is continuous, the I_{ξ_n} may be chosen so that the saltus of $\phi(x)$ in I_{ξ_n} is less than δ for every n . Consider the totality of intervals

$$I = [I_{\xi_n}], \quad \xi \in E - Z, n = 1, 2, \dots$$

By Vitali's covering theorem, since $m_e(E - Z) > 1 - \epsilon$, there is a finite number $I_{\xi_1 n_1}, I_{\xi_2 n_2}, \dots, I_{\xi_k n_k}$ of disjoint intervals of I the sum, $\sum_{i=1}^k m(I_{\xi_i n_i})$, of whose lengths is $1 - \alpha > 1 - \epsilon$. Let $G = I - \sum_{i=1}^k I_{\xi_i n_i}$ and let $G' \subset G$ be the points of G at which $f(x)$ is exteriorly approxi-

Presented to the Society, December 29, 1947; received by the editors August 20, 1947, and, in revised form, November 19, 1947.

¹ Numbers in brackets refer to the references cited at the end of the paper.

mately continuous. Every $\zeta \in G'$ is in a sequence $\{I_{\zeta m}\}$ of closed intervals contained in G , whose lengths converge to zero, such that for every m , the set of points x for which $|f(\zeta) - f(x)| < \delta$ has relative exterior measure in $I_{\zeta m}$ exceeding $1 - \delta/2$. Consider the totality of intervals

$$J = [I_{\zeta m}], \quad \zeta \in G', m = 1, 2, \dots$$

There is then a finite number of disjoint intervals $I_{\zeta_1 m_1}, I_{\zeta_2 m_2}, \dots, I_{\zeta_K m_K}$ of J such that $\sum_{i=1}^K m(I_{\zeta_i m_i})$ exceeds $\alpha - \delta/2$. Now, let

$$\begin{aligned} \psi(x) &= f(\xi_i) && \text{for every } x \in I_{\xi_i n_i}, i = 1, 2, \dots, k, \\ \psi(x) &= f(\zeta_i) && \text{for every } x \in I_{\zeta_i m_i}, i = 1, 2, \dots, K, \end{aligned}$$

and let $\psi(x)$ be defined elsewhere so as to be continuous. Suppose $x \in I_{\xi_i n_i}, 1 \leq i \leq k$.

$$\begin{aligned} |\psi(x) - \phi(x)| &\leq |\psi(x) - f(\xi_i)| + |f(\xi_i) - \phi(\xi_i)| \\ &\quad + |\phi(\xi_i) - \phi(x)| < 0 + (\epsilon - \delta) + \delta = \epsilon. \end{aligned}$$

But $\sum_{i=1}^k m(I_{\xi_i n_i}) > 1 - \epsilon$. Hence $|\psi(x) - \phi(x)| < \epsilon$ on a set of measure greater than $1 - \epsilon$. Moreover, since the relative exterior measure of the set of points for which $|\psi(x) - f(x)| < \delta < \eta$ exceeds $1 - \delta/2$ in every $I_{\xi_i n_i}, i = 1, 2, \dots, k$, and in every $I_{\zeta_i m_i}, i = 1, 2, \dots, K$, and these intervals are disjoint, the exterior measure of this set exceeds

$$\begin{aligned} \left(1 - \frac{\delta}{2}\right) &\left[\sum_{i=1}^k m(I_{\xi_i n_i}) + \sum_{i=1}^K m(I_{\zeta_i m_i}) \right] \\ &\geq \left(1 - \frac{\delta}{2}\right) \left[(1 - \alpha) + \left(\alpha - \frac{\delta}{2}\right) \right] \\ &= \left(1 - \frac{\delta}{2}\right)^2 > 1 - \delta > 1 - \eta. \end{aligned}$$

This completes the proof.

We now prove the theorem of Saks and Sierpinski. Let $f(x)$ be an arbitrary function defined on $I = [0, 1]$. Let $\phi_0(x) \equiv 0$. By the lemma, there is a continuous $\phi_1(x)$ such that $|\phi_1(x) - f(x)| < 1/2$ on a set of exterior measure greater than $1/2$. Having defined the continuous functions

$$\phi_1(x), \phi_2(x), \dots, \phi_{n-1}(x)$$

with $|\phi_{n-1}(x) - f(x)| < 1/2^{n-1}$ on a set of exterior measure greater than $1 - 1/2^{n-1}$ there is, by the lemma, a continuous $\phi_n(x)$ such that $|\phi_n(x) - \phi_{n-1}(x)| < 1/2^{n-1}$ on a set of measure greater than $1 - 1/2^{n-1}$

and $|\phi_n(x) - f(x)| < 1/2^n$ on a set of exterior measure greater than $1 - 1/2^n$. Let

$$\phi(x) = \limsup_{n \rightarrow \infty} \phi_n(x).$$

Then $\phi(x) = \lim_{n \rightarrow \infty} [\text{l.u.b. } (\phi_n(x), \phi_{n+1}(x), \dots)]$, and, since l.u.b. $(\phi_n(x), \phi_{n+1}(x), \dots)$ is lower semi-continuous, for every n , $\phi(x)$ is the limit of a nonincreasing sequence of lower semi-continuous functions and is, therefore, of Baire class 2 at most. Moreover, the sequence $\{\phi_n(x)\}$ itself converges almost everywhere to $\phi(x)$. Accordingly, for every $\epsilon > 0$ and every $\eta > 0$, there is an n , such that $|\phi_n(x) - \phi(x)| < \epsilon/2$ on a set of measure greater than $1 - \eta/2$ and $|\phi_n(x) - f(x)| < \epsilon/2$ on a set of exterior measure greater than $1 - \eta/2$. Hence, the set of points for which $|\phi(x) - f(x)| < \epsilon$ is of exterior measure greater than $1 - \eta$ for every $\eta > 0$ and is, therefore, of exterior measure one.

REFERENCES

1. S. Saks and W. Sierpinski, *Sur une propriété générale de fonctions*, Fund. Math. vol. 11 (1928) pp. 105-112.
2. H. Blumberg, *The measurable boundaries of an arbitrary function*, Acta Math. vol. 65 (1935) p. 277.

UNIVERSITY OF OKLAHOMA