

## ON THE LOCATION OF THE ZEROS OF THE DERIVATIVES OF A POLYNOMIAL SYMMETRIC IN THE ORIGIN

J. L. WALSH

If the zeros of a polynomial  $p(z)$  when plotted in the  $z$ -plane are symmetric in  $0: z=0$ , the zeros of the derivative  $p'(z)$  of  $p(z)$  can profitably be studied by transforming onto the  $w$ -plane, with  $w=z^2$ , and applying known theorems there.<sup>1</sup> It is the purpose of the present note to carry that study somewhat farther than has been previously done, in particular to consider the higher derivatives of  $p(z)$ .

Under the transformation  $w=u+iv=z^2=(x+iy)^2$ , an arbitrary line  $Au+Bv+C=0$  in the  $w$ -plane corresponds to an equilateral hyperbola  $A(x^2-y^2)+2Bxy+C=0$  in the  $z$ -plane with center  $0$  or to two perpendicular lines intersecting at  $0$ . A half-plane in the  $w$ -plane for which  $w=0$  is an interior or exterior point corresponds in the  $z$ -plane respectively to the exterior or interior of an equilateral hyperbola whose center is  $0$ ; a half-plane for which  $w=0$  is a boundary point corresponds to a double sector with vertex  $z=0$  and angle  $\pi/2$ . A point  $z$  is considered to be exterior or interior to a hyperbola according as the curve at its nearest point is convex or concave toward  $z$ .

We write the given polynomial in the form

$$(1) \quad p(z) = z^i \prod_{j=1}^q (z^2 - \alpha_j^2), \quad \alpha_j \neq 0,$$

and in the  $w$ -plane study the polynomials ( $w=z^2$ )

$$(2) \quad P(w) = P(z^2) = [p(z)]^2, \quad P'(w) = p(z) \cdot p'(z)/z.$$

Each zero of  $P(w)$  corresponds to a zero of  $p(z)$  and reciprocally; each zero of  $P'(w)$  corresponds to a zero of  $p(z)$  or  $p'(z)$  and reciprocally except that  $z=0$  is a zero of  $p'(z)$  unless  $z=0$  is a simple zero of  $p(z)$ .

We have (loc. cit.) by Lucas' Theorem

**THEOREM 1.** *If the zeros of  $p(z)$  are symmetric in  $0$  and lie in the closed exterior of an equilateral hyperbola with center  $0$  or in the closed exterior of a double sector with vertex  $0$  and angle  $\pi/2$ , then the zeros of  $p'(z)$  lie also in that closed exterior.*

*If the zeros of  $p(z)$  are symmetric in  $0$  and lie in the closed interior of an equilateral hyperbola with center  $0$ , then the zeros of  $p'(z)$  also lie in that closed interior except for a simple zero at  $0$ .*

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<sup>1</sup> Walsh, *Mathematica* vol. 8 (1933) pp. 185-190.

To the polynomial  $P(w)$  we now apply the theorem:<sup>2</sup>

*Let  $P(w)$  be a polynomial in  $w$  of degree  $n$  with an  $l$ -fold ( $l > 0$ ) zero at  $w=0$  and all the remaining zeros of  $P(w)$  in the closed half-plane  $\Pi$  not containing  $w=0$ . Then except for a zero at  $w=0$  of multiplicity  $l-1$ , all zeros of  $P'(w)$  lie in the closed half-plane obtained by shrinking  $\Pi$  toward the origin in the ratio  $n:l$ .*

The degrees of  $p(z)$  and  $P(w)$  defined by (1) and (2) are equal, as are the multiplicities of their zeros  $z=0$  and  $w=0$ . Under the transformation  $w=z^2$ , the line  $u=a$  corresponds to the hyperbola  $x^2-y^2=a$ , and the line  $u=la/n$  corresponds to  $x^2-y^2=la/n$ , so we have

**THEOREM 2.** *Let  $p(z)$  be a polynomial of degree  $n$  whose zeros are symmetric in the origin 0, let 0 be an  $l$ -fold zero ( $l > 0$ ), and let all the other zeros lie in the closed interior of an equilateral hyperbola  $H$  with center 0. Then except for an  $(l-1)$ -fold zero at 0, all zeros of  $p'(z)$  lie in the closed interior of the hyperbola obtained by shrinking  $H$  toward 0 in the ratio  $(n:l)^{1/2}$ .*

We turn now to the higher derivatives of the polynomial  $p(z)$ . Under the conditions of the first part of Theorem 1, the zeros of every derivative of  $p(z)$  not vanishing identically lie in the given closed region.

Suppose, however, the zeros of the polynomial  $p(z)$  of degree  $n$  lie in the closed interior of an equilateral hyperbola  $H$  whose center is 0. The polynomial  $p'(z)$  has a simple zero at 0, and by Theorem 1 the remaining zeros of  $p'(z)$  lie in the closed interior of  $H$ . By Theorem 2, all zeros of  $p''(z)$  lie in the closed interior of the hyperbola obtained by shrinking  $H$  toward 0 in the ratio  $[(n-1):1]^{1/2}$ . The higher derivatives of  $p(z)$  not vanishing identically have alternately a simple zero at 0 and no zero at 0. Continued application of Theorems 1 and 2, in alternation, then yields further hyperbolas which respectively contain the zeros other than 0 of the  $k$ th derivative of  $p(z)$ . However, except in the cases  $k=1$ , and  $k=2$  with  $n \geq 4$ , these hyperbolas are not the most favorable that can be obtained; a zero of  $p'(z)$  in Theorem 2 cannot lie on the new hyperbola unless all zeros of  $p(z)$  lie on  $H$  in two points symmetric in 0. We proceed to prove the following generalization of Theorem 2, the principal result of the present note:

**THEOREM 3.** *Let  $p(z)$  be a polynomial of degree  $n$  whose zeros are symmetric in 0, let 0 be an  $l$ -fold zero, and let the remaining zeros lie in the closed interior of a hyperbola  $H$  whose center is 0. Let  $p_0(z)$  be the polynomial of degree  $n$  whose zeros are symmetric in 0, which has an  $l$ -fold*

<sup>2</sup> Walsh, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 101-116; p. 115.

zero at 0, and whose remaining zeros lie at the vertices of  $H$ . Let  $z_k$ ,  $1 \leq k \leq n-2$ , be one of the zeros of  $p_0^{(k)}(z)$  of smallest positive modulus, and let  $H_k$  be the equilateral hyperbola with center 0 which has  $z_k$  as a vertex. Then all zeros of  $p^{(k)}(z)$  other than 0 lie in the closed interior of  $H_k$ .

Of course 0 need not be a zero of  $p^{(k)}(z)$ ; whether or not 0 is such a zero depends on  $l$ ,  $n$ , and  $k$ . To prove Theorem 3 we need a more powerful result than that used in proving Theorem 2:

Let  $Q(w)$  be a polynomial in  $w$  of degree  $q$ , and let the numbers  $A_j$  be constants. If the locus of the zeros of  $Q(w)$  is a closed half-plane  $\Pi$ , then the locus of the zeros of the polynomial

$$(3) \quad A_0 w^r Q(w) + A_1 w^{r+1} Q'(w) + \dots + A_q w^{r+q} Q^{(q)}(w)$$

is also a number of half-planes, identical with the locus of the zeros of the polynomial (3) with  $Q(w)$  replaced by  $(w-\alpha)^q$ :

$$(4) \quad A_0 w^r (w-\alpha)^q + q A_1 w^{r+1} (w-\alpha)^{q-1} + \dots + q(q-1) \dots \cdot 1 A_q w^{r+q}$$

when the locus of  $\alpha$  is  $\Pi$ .

The theorem just quoted is essentially a special case of a more general theorem<sup>3</sup> concerning a linear combination of products of derivatives of two polynomials. If polynomial (4) is written as the product of  $w^{r+q}$  and a polynomial in  $W = (w-\alpha)/w$  whose zeros are  $W = W_1, W_2, \dots, W_q$ , it is seen that the common locus of the zeros of (3) and (4) consists of the origin  $w=0$  (provided  $w=0$  is a zero of (4)) plus a number of half-planes, loci of the points  $w = \alpha/(1 - W_j)$ ,  $j=1, 2, \dots, q$ , when the locus of  $\alpha$  is  $\Pi$ ; if  $\Pi$  does not contain the point  $w=0$ , a zero  $W_j=1$  is to be ignored; if  $\Pi$  contains the point  $w=0$  and if  $W_j=1$  is a zero of (4), the entire plane is to be considered the locus of the zeros of (3) and (4). If  $\Pi$  does not contain 0, and if for real positive  $\alpha$  all zeros of (4) lie in the interval  $0 \leq w \leq \alpha$ , then the locus of the zeros of (4) is the origin (if  $w=0$  is a zero of (4)) plus a half-plane not containing 0 bounded by a line parallel to the boundary of  $\Pi$ , traced by the zero of (4) nearest to but different from 0 when  $\alpha$  traces the boundary of  $\Pi$ .

In Theorem 3 we write

$$p(z) = z^l p_1(z^2), \quad p_1(0) \neq 0, \\ p'(z) = lz^{l-1} p_1(z^2) + 2z^{l+1} p_1'(z^2),$$

<sup>3</sup> Walsh, Trans. Amer. Math. Soc. vol. 24 (1922) pp. 163-180; Theorem 9 of that paper applies to two circular regions, here specialized to a single point and a half-plane respectively.

$$p''(z) = l(l - 1)z^{l-2}p_1(z^2) + (4l + 2)z^l p_1'(z^2) + 4z^{l+2} p_1''(z^2),$$

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Except perhaps for a factor  $z$ , these derivatives are precisely of the form (3), with  $w = z^2$ . If we have  $p_0(z) = z^l(z^2 - \beta^2)^{(n-l)/2}$ , the zeros of  $p_0^{(k)}(z)$  lie on the line segment joining  $\beta$  and  $-\beta$ , and the corresponding zeros in the  $w$ -plane lie on the segment joining 0 and  $\beta^2$ ; the images in the  $w$ -plane of the zeros of  $p_1(z)$  lie in a half-plane  $\Pi$  not containing  $w = 0$ ; thus by the theorem quoted the images in the  $w$ -plane of the zeros of  $p^{(k)}(z)$  other than  $z = 0$  lie in the half-plane whose boundary is parallel to that of  $\Pi$ , which contains  $\Pi$  in its interior, and whose boundary passes through the image of the zero of  $p_0^{(k)}(z)$  nearest to but distinct from 0. Theorem 3 follows.

When as in Theorem 1 we study the zeros of a polynomial  $p(z)$  that are symmetric in 0, equilateral hyperbolas with center 0 are both (1) the lines of force in the  $z$ -plane in the Gaussian field due to two particles symmetric in 0 and (2) the images in the  $z$ -plane of the straight lines in the  $w$ -plane under the transformation  $w = z^2$ . Indeed, the lines of force due to particles at  $z = \alpha$  and  $z = -\alpha$  are loci  $\arg(z^2 - \alpha^2) = \text{const.}$ , arcs of equilateral hyperbolas with center 0; half-lines in the  $w$ -plane can be written  $\arg(w - w_0) = \text{const.}$ , and their images in the  $z$ -plane are loci  $\arg(z^2 - w_0) = \text{const.}$

A set of points in the  $z$ -plane is said to have  $m$ -fold symmetry about 0 if the set is unchanged by a rotation about 0 through an angle of  $2\pi/m$ . A polynomial  $p(z)$  whose zeros possess  $m$ -fold symmetry about 0 is readily studied by means of the transformation  $w = z^m$ . Under this transformation a straight line in the  $w$ -plane not through  $w = 0$  corresponds to a curve which may be called an  $m$ -hyperbola with center 0. A closed half-plane in the  $w$ -plane containing 0 in its interior corresponds to the closed exterior of an  $m$ -hyperbola, and a closed half-plane not containing 0 corresponds to the closed interior of an  $m$ -hyperbola. A line through  $w = 0$  corresponds to  $m$  equally spaced lines through  $z = 0$ , and a half-plane in the  $w$ -plane bounded by a line through  $w = 0$  corresponds to  $m$  equally spaced sectors in the  $z$ -plane, each of angle  $\pi/m$ . The function  $[p(w^{1/m})]^m$  is a polynomial in  $w$  which is readily studied in the  $w$ -plane, and yields results on the zeros of  $p'(z)$  in the  $z$ -plane. Theorem 3 extends directly to a polynomial  $p(z)$  whose zeros possess  $m$ -fold symmetry in 0. The  $m$ -hyperbolas with center 0 are not merely the images in the  $z$ -plane of the straight lines in the  $w$ -plane, but also the lines of force in the  $z$ -plane due to  $m$  particles  $m$ -fold symmetric in  $z = 0$ .