A NOTE ON THE OPERATORS OF BLASCHKE AND PRIVALOFF

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Let $f(P)$ be a function of a point $P \equiv P(x, y)$ in Euclidean 2-space. Let $L(f; P; r)$, $A(f; P; r)$ be the mean values of $f(P)$ on the perimeter and on the interior, respectively, of a circle of center P and radius r, that is,

$$L(f; P; r) = \frac{1}{2\pi r} \int_{C(P;r)} f(Q) ds_Q,$$

$$A(f; P; r) = \frac{1}{\pi r^2} \iint_{D(P;r)} f(Q) dQ$$

where $C(P; r)$, $D(P; r)$ are the perimeter and interior, respectively, of the circle with center P and radius r. The operators

$$\nabla_p f(P) = \lim_{r \to 0} \frac{4}{r^2} [L(f; P; r) - f(P)],$$

$$\nabla_a f(P) = \lim_{r \to 0} \frac{8}{r^2} [A(f; P; r) - f(P)]$$

have been defined by Blaschke and Privaloff, respectively. The following are a few of the results which have been obtained by these and other investigators.

THEOREM A [1, 2].[1] *If $f(P)$ has continuous second partial derivatives, then $\nabla_p f(P)$, $\nabla_a f(P)$ exist, and*

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)_P \equiv \nabla^2 f(P) = \nabla_p f(P) = \nabla_a f(P).$$

THEOREM B [1]. *If* (i) *$f(P)$ is continuous on a circle $\overline{D}(Q; r)$,* (ii) *$\nabla_p f(P)$ exists on the interior, $D(Q; r)$, then*

$$\frac{4}{r^2} [L(f; Q; r) - f(Q)]$$

lies between the upper and lower bounds of $\nabla_p f(P)$ on $D(Q; r)$.

THEOREM C [3, 4]. *If $u(P)$ is a logarithmic potential function*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$u(P) = \int_W \log \frac{1}{PQ} \, d\mu(Q)$$

where μ *is a mass distribution, and if the density exists at* R, *that is,*

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{D(R;r)} d\mu(Q) = D_s\mu(R)$$

exists, then $\nabla_p u(R)$, $\nabla_a u(R)$ *exist and* $\nabla_p u(R) = \nabla_a u(R) = -2\pi D_s\mu(R)$. *(*$W$ *indicates integration over the whole space.)*

The purpose of this note is to give extensions of Theorems B and C. Theorem B is readily extended to the operator ∇_a by the following:

THEOREM 1. *If* (i) $f(P)$ *is continuous on a circle* $\overline{D}(Q; r)$, (ii) $\nabla_a f(P)$ *exists on the interior,* $D(Q; r)$, *then*

$$\frac{8}{r^2} [A(f; Q; r) - f(Q)]$$

lies between the upper and lower bounds of $\nabla_a f(P)$ *on* $D(Q; r)$.

PROOF. Consider the function

$$\lambda(P) = f(P) - h(P) + L(f; Q; \rho) - f(Q) - \frac{1}{\rho^2} [L(f; Q; \rho) - f(Q)] \overline{PQ}^2$$

where $\rho \leqq r$, and $h(P)$ is the function harmonic on $D(Q; \rho)$ and such that $h(P) = f(P)$ on $C(Q; \rho)$. Clearly $\lambda(P) = 0$ on $C(Q; \rho)$. Further $\lambda(Q) = L(f; Q; \rho) - h(Q)$. But

$$h(Q) = \frac{1}{2\pi r} \int_{C(Q;\rho)} h(P) ds_P = \frac{1}{2\pi r} \int_{C(Q;\rho)} f(P) ds_P = L(f; Q; \rho).$$

Therefore $\lambda(Q) = 0$. Thus the continuous function $\lambda(P)$ has both a maximum and a minimum value on $D(Q; \rho)$. Now if R is a maximum point of $\lambda(P)$ then $\nabla_a \lambda(R) \leqq 0$, for

$$\nabla_a \lambda(R) = \lim_{\rho \to 0} \frac{1}{\pi \rho^4} \iint_{D(P;\rho)} [\lambda(P) - \lambda(R)] dP \leqq 0.$$

But

$$\nabla_a \lambda(P) = \nabla_a f(P) - \nabla_a h(P) - \frac{1}{\rho^2} [L(f; Q; \rho) - f(Q)] \nabla_a \overline{PQ}^2.$$

By Theorem A, $\nabla_a h(P) = 0$, $\nabla_a \overline{PQ}^2 = 4$. Therefore

$$\nabla_a \lambda(P) = \nabla_a f(P) - \frac{4}{\rho^2}\,[L(f; Q; \rho) - f(Q)].$$

But $\nabla_a \lambda(R) \leqq 0$, hence

$$\nabla_a f(R) \leqq \frac{4}{\rho^2}\,[L(f; Q; \rho) - f(Q)].$$

Similarly if S is a minimum point of $\lambda(P)$ on $D(Q; \rho)$ we have

$$\nabla_a f(S) \geqq \frac{4}{\rho^2}\,[L(f; Q; \rho) - f(Q)].$$

Thus, if M, m are the upper and lower bounds, respectively, of $\nabla_a f(P)$ on $D(Q; r)$, then for all $\rho \leqq r$

$$m \leqq \frac{4}{\rho^2}\,[L(f; Q; \rho) - f(Q)] \leqq M,$$

and so

$$\frac{2}{r^2}\int_0^r m\rho^3 d\rho \leqq \frac{8}{r^2}\int_0^r L(f; Q; \rho)\rho d\rho - \frac{8}{r^2}\int_0^r f(Q)\rho d\rho \leqq \frac{2}{r^2}\int_0^r M\rho^3 d\rho.$$

But

$$A(f; Q; r) = \frac{1}{\pi r^2}\iint_{D(Q;r)} f(P)dP = \frac{2}{r^2}\int_0^r \rho d\rho \cdot \frac{1}{2\pi\rho}\int_{C(Q;\rho)} f(P)ds_P$$
$$= \frac{2}{r^2}\int_0^r L(f; Q; \rho)\rho d\rho.$$

Thus $mr^2/2 \leqq 4[A(f; Q; r) - f(Q)] \leqq Mr^2/2$ and

$$m \leqq \frac{8}{r^2}\,[A(f; Q; r) - f(Q)] \leqq M.$$

For the operator ∇_a a somewhat stronger form of Theorem C is obtainable.

THEOREM 2. *If $u(P)$ is a logarithmic potential function*

$$u(P) = \int_W \log \frac{1}{PQ}\, d\mu(Q)$$

where μ is a mass distribution, and if at R

$$\lim_{r\to 0} \operatorname{ap} \frac{1}{\pi r^2}\int_{D(R;r)} d\mu(P) = D_a\mu(R)$$

exists, then $\nabla_a u(R)$ *exists, and* $\nabla_a u(R) = -2\pi D_a\mu(R)$.

PROOF. Consider

$$L(u; R; \rho) = \frac{1}{2\pi\rho}\int_{C(R;\rho)} u(P)ds_P = \frac{1}{2\pi\rho}\int_{C(R;\rho)} ds_P\cdot\int_W \log\frac{1}{PQ}\,d\mu(Q)$$

$$= \int_W d\mu(Q)\cdot\frac{1}{2\pi\rho}\int_{C(R;\rho)} \log\frac{1}{PQ}\,ds_P.$$

Now

$$\frac{1}{2\pi\rho}\int_{C(R;\rho)} \log\frac{1}{PQ}\,ds_P = \log\frac{1}{QR} \qquad (QR > \rho)$$

$$= \log\frac{1}{\rho} \qquad (QR \leqq \rho).$$

Hence

$$L(u; R; \rho) = \int_{D(R;\rho)} d\mu(Q)\cdot\log\frac{1}{\rho} + \int_{W-D} d\mu(Q)\cdot\log\frac{1}{QR}$$

$$= \int_W d\mu(Q)\cdot\log\frac{1}{QR} + \int_{D(R;\rho)}\left[\log\frac{1}{\rho} - \log\frac{1}{QR}\right]d\mu(Q)$$

$$= u(R) + \int_{D(R;\rho)}\left[\log\frac{1}{\rho} - \log\frac{1}{QR}\right]d\mu(Q).$$

Thus

$$A(u; R; r) = \frac{2}{r^2}\int_0^r L(u; R; \rho)\rho d\rho$$

$$= u(R) + \frac{2}{r^2}\int_0^r \rho d\rho\cdot\int_{D(R;\rho)}\left[\log\frac{1}{\rho} - \log\frac{1}{QR}\right]d\mu(Q),$$

and so

$$\frac{8}{r^2}\left[A(u; R; r) - u(R)\right]$$

$$= \frac{16}{r^4}\int_0^r \rho d\rho\cdot\int_{D(R;\rho)}\left[\log\frac{1}{\rho} - \log\frac{1}{QR}\right]d\mu(Q).$$

The integrand depends only on $|QR|$, so we can write

$$\frac{8}{r^2}\left[A(u;R;r)-u(R)\right]=\frac{16}{r^4}\int_0^r \rho d\rho\cdot\int_0^\rho\left[\log\frac{1}{\rho}-\log\frac{1}{t}\right]d\bar{\mu}(t)$$

where

$$\bar{\mu}(t)=\int_{D(R;t)}d\mu(Q).$$

Integrating by parts we have

$$\frac{8}{r^2}\left[A(u;R;r)-u(R)\right]$$
$$=\frac{16}{r^4}\int_0^r \rho d\rho\cdot\left\{\left[\left(\log\frac{1}{\rho}-\log\frac{1}{t}\right)\bar{\mu}(t)\right]_0^\rho-\int_0^\rho \bar{\mu}(t)\frac{dt}{t}\right\}.$$

But

$$\bar{\mu}(t)=\int_{D(R;t)}d\mu(Q)=\pi t^2 D_a\mu(R)+o(t^2)$$

for almost all small t. Hence

$$\frac{8}{r^2}\left[A(u;R;r)-u(R)\right]$$
$$=\frac{16}{r^4}\int_0^r \rho d\rho\cdot\left\{-\int_0^\rho \pi t^2 D_a\mu(R)\frac{dt}{t}-\int_0^\rho o(t^2)\frac{dt}{t}\right\}$$
$$=\frac{16}{r^4}\int_0^r \rho d\rho\cdot\left[-\frac{\pi}{2}\rho^2 D_a\mu(R)+o(\rho^2)\right]$$
$$=-2\pi D_a\mu(R)+\frac{16}{r^4}\int_0^r o(\rho^3)d\rho.$$

Thus

$$\lim_{r\to 0}\frac{8}{r^2}\left[A(u;R;r)-u(R)\right]=\nabla_a u(R)=-2\pi D_a\mu(R).$$

Many results which have been proven for one operator can be extended to the other operator by use of the following theorem.

THEOREM 3. *If $\nabla_p f(P)$ exists, then so does $\nabla_a f(P)$, and $\nabla_a f(P)=\nabla_p f(P)$.*

PROOF. $L(f; P; r)$ exists for small r, and further

$$L(f; P; r) = f(P) + \frac{r^2}{4} \nabla_p f(P) + o(r^2).$$

Also

$$A(f; P; r) = \frac{2}{r^2} \int_0^r L(f; P; \rho)\rho d\rho.$$

And hence

$$\begin{aligned}
\frac{8}{r^2} &[A(f; P; r) - f(P)] \\
&= \frac{8}{r^4}\left[2 \int_0^r L(f; P; \rho)\rho d\rho - r^2 f(P)\right] \\
&= \frac{8}{r^4}\left\{2 \int_0^r \left[f(P) + \frac{\rho^2}{4} \nabla_p f(P) + o(\rho^2)\right] \rho d\rho - r^2 f(P)\right\} \\
&= \nabla_p f(P) + \frac{16}{r^4} \int_0^r o(\rho^3) d\rho.
\end{aligned}$$

The last term is easily seen to approach zero as $r \to 0$. Thus

$$\lim_{r \to 0} \frac{8}{r^2} [A(f; P; r) - f(P)] = \nabla_a f(P) = \nabla_p f(P).$$

These results hold true for spaces of higher dimensions, and the above proofs follow through with obvious modifications of the coefficients and the form of the potential function.

BIBLIOGRAPHY

1. W. Blaschke, *Ein Mittelwertsatz und eine kennzeichende Eigenschaft des logarithmischen Potentials*, Berichte über die Verhandlungen der königlich sachsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Klasse vol. 68 (1916) pp. 3–7..

2. I. Privaloff, *Sur les functions harmonique*, Rec. Math. (Mat. Sbornik) vol. 32 (1924–1925) pp. 464–469.

3. ———, *On a theorem of S. Saks*, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 457–460.

4. S. Saks, *On the operators of Blaschke and Privaloff for subharmonic functions*, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 451–456.

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