A NOTE ON LACUNARY POLYNOMIALS

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1. **Introduction.** In the present note we shall give an elementary derivation of some new bounds for the p smallest (in modulus) zeros of the polynomials of the lacunary type

$$(1.1)\quad f(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots + a_{n_k} z^{n_k},$$
$$a_0 a_p \neq 0,\ 0 < p = n_0 < n_1 < \cdots < n_k.$$

This will be done by the iterated application, first, of Kakeya's Theorem[1] that, if a polynomial of degree n has p zeros in a circle C of radius R, its derivative has at least $p-1$ zeros in the concentric circle C' of radius $R' = R\phi(n, p)$; and, secondly, of the specific limits

$$(1.2)\qquad \phi(n, p) \leqq \csc\,[\pi/2(n - p + 1)],$$

$$(1.3)\qquad \phi(n, p) \leqq \prod_{j=1}^{n-p} (n + j)/(n - j)$$

furnished by Marden[2] and Biernacki[3] respectively.

2. **Derivation of the bounds.** An immediate corollary to Kakeya's Theorem is:

THEOREM I. *If the derivative of an nth degree polynomial $P(z)$ has at most $p-1$ zeros in a circle Γ of radius ρ, then $P(z)$ has at most p zeros in the concentric circle Γ' of radius $\rho' = \rho/\phi(n, p+1)$.*

We shall use Theorem I to prove the following theorem.

THEOREM II. *If all the zeros of the polynomial*

$$(2.1)\quad \begin{aligned} f_0(z) = {} & n_1 n_2 \cdots n_k a_0 + (n_1 - 1)(n_2 - 1) \cdots (n_k - 1) a_1 z \\ & + \cdots + (n_1 - p)(n_2 - p) \cdots (n_k - p) a_p z^p \end{aligned}$$

lie in the circle $|z| \leqq R_0$, at least p zeros of polynomial (1.1) *lie in the circle*

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[1] S. Kakeya, Tôhoku Math. J. vol. 11 (1917) pp. 5–16.

[2] M. Marden, Trans. Amer. Math. Soc. vol. 45 (1939) pp. 335–368. See also M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, to be published as a volume of Mathematical Surveys.

[3] M. Biernacki, Bull. Soc. Math. France (2) vol. 69 (1945) pp. 197–203.

$$|z| \leqq R(p, k) = R_0 \prod_{i=1}^{k} \phi(n_i, n_i - p + 1).$$

For this purpose we define the sequence of polynomials

$$F_0(z) \equiv z^{n_k} f(1/z), \tag{2.2}$$

$$F_j(z) \equiv z^{1-n_{k-j+1}+n_{k-j}} F'_{j-1}(z), \qquad j = 1, 2, \cdots, k. \tag{2.3}$$

We may verify easily that

$$F_k(z) = z^p f_0(1/z). \tag{2.4}$$

All the zeros of $F_k(z)$ therefore lie outside the circle $|z| \geqq (1/R_0)$. By equation (2.3), the zeros of $F'_{k-1}(z)$ are the zeros of $F_k(z)$ and a zero of multiplicity $n_1 - p - 1$ at the origin and, hence, only the latter lies inside $|z| < 1/R_0$. By Theorem I, $F_{k-1}(z)$ has at most $n_1 - p$ zeros in

$$|z| < [R_0 \phi(n_1, n_1 - p + 1)]^{-1} = 1/R(p, 1).$$

Let us now assume, as already verified for $j = 1, 2, \cdots, s$, that $F_{k-j}(z)$ has at most $n_j - p$ zeros in the circle $|z| < 1/R(p, j)$. From equations (2.3) with j replaced by $k - s$, it follows then that $F'_{k-s-1}(z)$ has zeros of total multiplicity at most

$$(n_{s+1} - n_s - 1) + (n_s - p) = n_{s+1} - p - 1$$

in this circle. By Theorem I, therefore, $F_{k-s-1}(z)$ has at most $n_{s+1} - p$ zeros in the circle

$$|z| < [R(p, s)\phi(n_{s+1}, n_{s+1} - p + 1)]^{-1} = 1/R(p, s + 1).$$

By mathematical induction, it follows that $F_0(z)$ has at most $n_k - p$ zeros in the circle $|z| < 1/R(p, k)$.

By (2.2), $f(z)$ has therefore at most $n_k - p$ zeros outside the circle $|z| = R(p, k)$ and hence at least p zeros in or on this circle.

By using the limits (1.2) and (1.3), we now deduce from Theorem II the following corollary.

COROLLARY 1. *At least p zeros of polynomial* (1.1) *lie in each of the circles*

$$|z| \leqq R_0 \csc^k (\pi/2p), \tag{2.5}$$

$$|z| \leqq R_0 \prod_{i=1}^{k} \prod_{j=1}^{p-1} (n_i + j)/(n_i - j). \tag{2.6}$$

If it is known that all the zeros of the polynomial

$$h(z) = a_0 + a_1 z + \cdots + a_p z^p$$

lie in the circle $|z| \leqq R_1$, then the application of a theorem in a previous paper[4] permits us to take

$$R_0 \leqq [R_1 n_1 n_2 \cdots n_k/(n_1 - p)(n_2 - p) \cdots (n_k - p)] = R_2.$$

As (2.6) with R_0 replaced by R_2 is the bound furnished recently by Biernacki,[3] we see that the bound (2.6) is at least as good as his bound.

3. **Application to lacunary series.** We shall now use Corollary 1 to prove the following theorem.

THEOREM III. *Let ρ_1, $0<\rho_1 \leqq \infty$, be the radius of convergence of the series*

$$g(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots,$$
$$a_0 a_p \neq 0, \qquad 1 \leqq p < n_1 < n_2 < \cdots.$$

Let the series $\sum(1/n_j)$ be convergent, so that the product

$$A(m) = \prod_{j=1}^{\infty} [1 - (m/n_j)]$$

is also convergent. Let ρ, the radius of the circle $|z| = \rho$ containing all the zeros of the polynomial

$$G(z) = A(0)a_0 + A(1)a_1 z + \cdots + A(p)a_p z^p,$$

be such that

$$\rho \prod_{j=1}^{p-1} A(-j)/A(j) = \rho_2 < \rho_1.$$

Then $g(z)$ has at least p zeros in the circle $|z| \leqq \rho_2$.

Let us consider equations (1.1) and (2.1) as defining the sequences of polynomials $f(z, k)$ and $f_0(z, k)$ respectively. When $k \to \infty$, the sequence $[f_0(z, k)/n_1 n_2 \cdots n_k]$ converges uniformly to $G(z)$ in $|z| \leqq \rho$. By Hurwitz' theorem, for any given positive ϵ, we may choose a positive k_1 so large that all the zeros of each $f_0(z, k)$, $k \geqq k_1$, lie in the circle $|z| \leqq \rho + \epsilon$. By Corollary 1, at least p zeros of the $f(z, k)$, $k \geqq k_1$, lie in the circle

$$|z| \leqq (\rho + \epsilon) \prod_{j=1}^{p-1} \prod_{i=1}^{k} (1 + j/n_i)/(1 - j/n_i) < \rho_2 + \epsilon\,(\rho_2/\rho) = \rho_2'.$$

[4] M. Marden, Bull. Amer. Math. Soc. vol. 49 (1943) p. 97, Corollary.

Choosing ϵ so small that $\rho_2' + \epsilon < \rho_1$, we see that the $f(z, k)$ converge uniformly to $g(z)$ in $|z| \leqq \rho_2'$. Thus $g(z)$ has p zeros in the circle $|z| < \rho_2' + \epsilon$ and, since ϵ is arbitrary, in the circle $|z| \leqq \rho_2$.

As a corollary to Theorem III, we may prove that, if $g(z)$ is an entire function, it assumes every finite value an infinite number of times.

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