ON SOME CRITERIA OF CARLEMAN FOR THE COMPLETE CONVERGENCE OF A J-FRACTION

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1. Introduction. Carleman [1, pp. 214–215][1] derived, from his theory of integral equations, a number of criteria for the complete convergence of a real J-fraction

$$(1.1) \qquad -\mathop{K}_{p=1}^{\infty} \frac{-a_{p-1}^{2}}{b_p + z} \qquad (a_0 = 1,\ a_p \neq 0).$$

The most important one of these criteria states that the J-fraction is completely convergent if the series $\sum(1/c_{2p})^{1/2p}$ diverges, where $c_0, c_1, c_2, \cdots$ are the coefficients in the power series $\sum(c_p/z^{p+1})$ associated with the J-fraction. In [2] Carleman gave an algebraic proof of this theorem for the case where $b_p = 0$, $p = 1, 2, 3, \cdots$. The present note contains an algebraic proof for the general case, and some remarks concerning the other criteria of Carleman, especially with reference to their application to J-fractions with arbitrary complex coefficients.

2. The determinate and indeterminate cases. Let $a_0 = 1$, $a_p \neq 0$, b_p, $p = 1, 2, 3, \cdots$, be complex constants, and consider the system of linear equations

$$(2.1) \qquad -a_{p-1}x_{p-1} + (b_p + z)x_p - a_p x_{p+1} = 0, \qquad p = 1, 2, 3, \cdots.$$

Since the a_p are not zero, these equations determine $x_2, x_3, x_4, \cdots$ uniquely in terms of arbitrarily chosen initial values x_0, x_1. If $x_0 = -1$, $x_1 = 0$, let $x_p = X_p(z)$, and if $x_0 = 0$, $x_1 = 1$, let $x_p = Y_p(z)$. Then, $X_{p+1}(z)/Y_{p+1}(z)$ is the pth approximant of the J-fraction (1.1). If the infinite series $\sum|X_p(z)|^2$ and $\sum|Y_p(z)|^2$ both converge for one value of z, then they converge for every value of z [7, p. 120]. We may accordingly distinguish two cases for a J-fraction (1.1) with complex coefficients. In the *indeterminate case*, both the infinite series

$$(2.2) \qquad \sum|X_p(0)|^2, \qquad \sum|Y_p(0)|^2$$

are convergent. In the *determinate case*, at least one of these infinite series is divergent. A real J-fraction is completely convergent if and only if the determinate case holds. This is also the case of a determinate moment problem [4].

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

On applying Schwarz's inequality to the determinant formulas

$$(2.3)\qquad X_{p+1}(z)Y_p(z) - X_p(z)Y_{p+1}(z) = \frac{1}{a_p}$$

and

$$(2.4)\qquad X_{p+2}(z)Y_p(z) - X_p(z)Y_{p+2}(z) = \frac{b_{p+1} + z}{a_p a_{p+1}},$$

after setting $z=0$, we find immediately that when the indeterminate case holds then the series $\sum(1/|a_p|)$ and $\sum|b_{p+1}/a_p a_{p+1}|$ are convergent. These and other criteria obtained by applying Schwarz's inequality to other determinant formulas were found by Dennis and Wall [3]. They were not aware of the fact that Carleman [1, p. 215] had given the first of these criteria for the case of real J-fractions. Hellinger [6] showed that the determinate case holds for a real J-fraction if lim inf $|a_p|$ is finite. This criterion, which is contained in the first criterion mentioned above, was given by Carleman [1, p. 215].

Let $\delta_1, \delta_2, \delta_3, \cdots$ be complex numbers such that $\delta \leqq |\delta_p| \leqq \Delta$, $p=1, 2, 3, \cdots$, where δ and Δ are positive constants independent of p. *If the determinate case holds for the J-fraction* (1.1), *then the determinate case holds for the J-fraction*

$$(2.5)\qquad -\mathop{K}_{p=1}^{\infty} \frac{-a_{p-1}^2\delta_{p-1}\delta_p}{b_p\delta_p + z} \qquad (\delta_0 = 1).$$

The introduction of the factors δ_k into (1.1) effects the replacement of the series (2.2) by the series

$$\sum \left|\frac{\delta_1}{\delta_p}\right| |X_p(0)|^2, \qquad \sum \left|\frac{\delta_1}{\delta_{p+1}}\right| |Y_p(0)|^2,$$

so that the theorem is obviously true. Carleman [1, p. 215] obtained this for the case of real a_p, b_p, δ_p as a corollary to his general theory.

Let α_p and β_p be complex numbers such that

$$|\alpha_p| < M, \quad |\beta_p| < M, \qquad a_p + \alpha_p \neq 0,\ p = 1, 2, 3, \cdots.$$

If the determinate case holds for the J-fraction (1.1), *then the determinate case holds for the J-fraction*

$$(2.6)\qquad -\mathop{K}_{p=1}^{\infty} \frac{-(a_{p-1}+\alpha_{p-1})^2}{b_p + \beta_p + z} \qquad (\alpha_0 = 0).$$

Carleman [1, p. 214] gave this theorem for the case of real a_k, b_k,

α_k, β_k. If these numbers are complex and $\alpha_k=0$, $k=1, 2, 3, \cdots$, then the theorem can be proved by means of an easy modification of the proof given in [7, pp. 120–121]. (Cf. also [8, p. 557].) The method used in [7] can be modified to cover the case $\beta_k=0$, $k=1, 2, 3, \cdots$, and hence to establish the theorem in the general case. We shall indicate briefly those modifications. Let

$$L_p(x) = -a_{p-1}x_{p-1} + b_px_p - a_px_{p+1}, \qquad p = 1, 2, 3, \cdots,$$

and let $L_p'(x)$ denote the expression $L_p(x)$ in which a_p is replaced by $a_p+\alpha_p$, where the α_p are any complex constants such that $|\alpha_p|<M$, $a_p+\alpha_p\neq 0$, $p=1, 2, 3, \cdots$. The solution of the system $L_p(x)=0$ under the initial conditions $x_0=-1$, $x_1=0$ is $x_p=X_p(0)$, and under the initial conditions $x_0=0$, $x_1=1$ the solution is $x_p=Y_p(0)$. Let x_p' be the solution of the system $L_p'(x)=0$ under the initial conditions $x_0'=a$, $x_1'=b$, where a and b are arbitrarily chosen constants. The theorem will be established if we show that the convergence of the series $\sum|X_p(0)|^2$ and $\sum|Y_p(0)|^2$ implies the convergence of the series $\sum|x_p'|^2$.

If ξ_p and ξ_p' are arbitrary solutions of the systems $L_p=0$ and $L_p'=0$, respectively, then we have the Green's formula

$$\sum_{p=1}^{n}[\xi_pL_p'(\xi') - \xi_p'L_p(\xi)] = [(\xi_p\xi_{p+1}' - \xi_p'\xi_{p+1})a_p]_0^n \\ - \sum_{p=1}^{n}(\alpha_{p-1}\xi_{p-1}' + \alpha_p\xi_{p+1}')\xi_p = 0. \tag{2.7}$$

If, in particular, $\xi_p=X_p(0)$, $\xi_p'=x_p'$, this gives

$$-b + (x_n'X_{n+1} - x_{n+1}'X_n)a_n - \sum_{p=1}^{n}(\alpha_{p-1}x_{p-1}' + \alpha_px_{p+1}')X_p = 0,$$

where we have written X_k for $X_k(0)$; and if $\xi_p=Y_p(0)=Y_p$, $\xi_p'=x_p'$, we obtain

$$-a + (x_n'Y_{n+1} - x_{n+1}'Y_n)a_n - \sum_{p=1}^{n}(\alpha_{p-1}x_{p-1}' + \alpha_px_{p+1}')Y_p = 0.$$

On multiplying the first of these equations by Y_n, the second by $-X_n$, and adding, we find that

$$x_n' - \sum_{p=1}^{n}k_{n,p}x_p' = h_n,$$

where

$$k_{n,p} = \begin{cases} \dfrac{\alpha_p(X_{p+1}Y_n - X_nY_{p+1}) + \alpha_{p-1}(X_{p-1}Y_n - X_nY_{p-1})}{1 + (\alpha_{n-1}/a_{n-1})}, & p = 1, 2, \cdots, n-1, \\ 0, \quad p = n, \end{cases}$$

$$h_n = \frac{bY_n - aX_n}{1 + (\alpha_{n-1}/a_{n-1})}.$$

Under the hypothesis that the series $\sum|X_p|^2$ and $\sum|Y_p|^2$ converge, it follows, as indicated before, that the series $\sum|1/a_p|$ converges, so that $\lim_{n\to\infty}|a_{n-1}| = \infty$. Since, by hypothesis, $|\alpha_p| < M$, $p=1, 2, 3, \cdots$, we then find at once by Schwarz's inequality that the double series $\sum|k_{pq}|^2$ and the series $\sum|h_p|^2$ are convergent. It then follows exactly as in [7, p. 121] that the series $\sum|x_p'|^2$ is convergent, and the theorem is thereby established.

3. **Criterion involving the moments c_p.** Let $\sum(c_p/z^{p+1})$ be the power series expansion in descending powers of z of a real J-fraction (1.1). *If the series*

$$\sum_{p=1}^{\infty}\left(\frac{1}{c_{2p}}\right)^{1/2p} \tag{3.1}$$

diverges, then the determinate case holds for the J-fraction. We shall give a simple algebraic proof of this well known theorem of Carleman [1, p. 215].

The quadratic forms

$$F_n = \sum_{p,q=1}^{n} c_{p+q-2}x_px_q, \qquad n = 1, 2, 3, \cdots,$$

are all positive definite [4], and we may therefore write

$$F_n \equiv (b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n)^2 + (b_{22}x_2 + b_{23}x_3 + \cdots + b_{2n}x_n)^2 + \cdots + (b_{nn}x_n)^2,$$

where the b_{pq} are real constants independent of n for $p \leqq n$, $q \leqq n$, and where $b_{pp} \neq 0$, $p=1, 2, 3, \cdots, n$. The b_{pp} are connected with the partial numerators of the J-fraction by the formulas [9]

$$(a_0a_1 \cdots a_p)^2 = b_{p+1,p+1}^2 \qquad (a_0 = b_{11} = 1)$$

and, obviously,

$$c_{2p} = \sum_{q=1}^{p+1} b_{q,p+1}^2 \qquad (p = 0, 1, 2, \cdots)$$

so that $c_{2p} \geqq b_{p+1,p+1}^2$. Consequently,

$$(a_0 a_1 \cdots a_p)^2 \leqq c_{2p}.$$

Hence, by Carleman's inequality[2] [2],

$$e \sum_{p=1}^{n} \frac{1}{|a_p|} > \sum_{p=1}^{n} \left(\frac{1}{a_1^2 a_2^2 \cdots a_p^2} \right)^{1/2p} \geqq \sum_{p=1}^{n} \left(\frac{1}{c_{2p}} \right)^{1/2p}.$$

Therefore, if the series (3.1) is divergent, then the series $\sum |1/a_p|$ is divergent, so that, as indicated in §2, the determinate case holds for the J-fraction.

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[2] For an algebraic proof of Carleman's inequality see, for instance [5].