tially the two following properties of $\gamma(Z, W)$:

  (a) $\gamma(Z, W)$ has at $Z = W$ a logarithmic pole;

  (b) $\gamma(Z, W)$ is a symmetric harmonic function;

while the third and most restrictive property of $\gamma(Z, W)$, that is, that it vanishes on the boundary C of B, was only used in the transformation of (21) into (21'). It might be more convenient for practical computations to use instead of $\gamma(Z, W)$ simply the singularity function $-\log|z-w|$ which satisfies both conditions (a) and (b). The representation of the Neumann function is in this case a little more complicated, but on the other hand one is saved the effort of computing the Green's function of the domain with respect to Laplace's equation.

    Finally, we wish to point out that we dealt with a particular type of the partial differential equation (1) only for the sake of a simple representation of the method. We indicated in our previous paper the possible generalizations of the type of equation, and for all those partial differential equations obtained one may construct the Green and Neumann functions in an analogous way.

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THE ISOPERIMETRIC PROBLEM OF BOLZA
WITH FINITE SIDE CONDITIONS

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    1. Introduction. In view of recent improvements in the theory of the problem of Bolza in the calculus of variations, both with respect to the necessary $[2, 6]$[1] and the sufficient conditions $[3, 4, 5, 7]$, it seems worthwhile to investigate certain problems which, although not immediate special cases of the problem of Bolza, can be transformed by simple means into such problems. A comparatively simple problem of this nature is that of finding necessary conditions and sufficient conditions in order that an arc

$$C: \qquad a_h, \; y_i(x) \qquad (h = 1, \cdots, r; \; i = 1, \cdots, n; \; x_1 \leqq x \leqq x_2)$$

will minimize an expression of the form

$$(1.1) \qquad I(C) = g(a) + \int_{x_1}^{x_2} f(a, x, y, y') dx$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

in the class of arcs C which satisfy

(1.2) $\qquad\qquad x_s = x_s(a), \qquad y_i(x_s) = y_{is}(a) \qquad\qquad (s = 1, 2),$

(1.3) $\qquad \phi_\sigma(a, x, y, y') = 0 \qquad (\sigma = q + 1, \cdots, q + m < n),$

(1.4) $\quad g_\rho(a) + \displaystyle\int_{x_1}^{x_2} f_\rho(a, x, y, y')dx = 0 \quad (\rho = q + 1, \cdots, q + p),$

and in addition finite side conditions of the form

(1.5) $\qquad\qquad\qquad \psi_\mu(a, x, y) = 0 \qquad\qquad (\mu = 1, \cdots, q).$

It is assumed that the functions appearing in equations (1.1) to (1.5) are continuous and have continuous partial derivatives of the first three orders in a region R of points (a, x, y, y'). We suppose further that $x_1(a) < x_2(a)$ and that the matrix

(1.6) $\qquad\qquad\qquad\qquad \left\| \begin{matrix} \phi_{\sigma y_i'} \\ \psi_{\mu y_i} \end{matrix} \right\|$

has rank $m+q$ on R.

For the case in which the parameters a_h are fixed and the conditions (1.3) and (1.4) are absent, Bower [1] has given a treatment of this problem.[2] Her treatment, however, is complicated by the fact that the associated isoperimetric problem of Bolza to be introduced in the next section is abnormal. Since normality is no longer needed as a hypothesis in either the necessary or the sufficient conditions, we can give a much simpler treatment of her problem. Introducing the parameters a_h and the conditions (1.3) and (1.4) provides no essential difficulty.

2. **Reduction to an equivalent isoperimetric problem of Bolza.** Suppose that C is an arc which satisfies (1.2) and (1.5). Then C also satisfies

(2.1) $\qquad \phi_\mu(a, x, y, y') \equiv \psi_{\mu x}(a, x, y) + \psi_{\mu y_i}(a, x, y)y_i' = 0,$

(2.2) $\qquad\qquad g_{\mu s}(a) \equiv \psi_\mu[a, x_s(a), y_s(a)] = 0 \qquad \text{when } s = 1.$

Conversely, if C satisfies (1.2), (2.1) and (2.2), then it satisfies (1.5). Therefore, C furnishes a weak or strong relative minimum for $I(C)$ in the class of arcs satisfying (1.2), (1.3), (1.4) and (1.5) if and only if C furnishes a weak or strong relative minimum for $I(C)$ in the class of arcs satisfying (1.2), (1.3), (1.4), (2.1) and (2.2). Since conditions (2.1) are ordinary differential equations, and since conditions (2.2)

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[2] This paper also contains references to the earlier work of Mayer, Bolza, Hadamard and Bliss.

may be regarded as additional isoperimetric conditions, this second problem is an ordinary isoperimetric problem of Bolza.

3. **The necessary conditions.** Suppose that C is an arc of class C'' which minimizes $I(C)$ in the class of arcs satisfying (1.2), (1.3), (1.4) and (1.5). Then we can prove that for each variation

$$\gamma: \qquad \alpha_h, \qquad \eta_i(x)$$

which satisfies

(3.1) $$\eta_i(x_s) = [y_{ish} - y_i'(x_s)x_{sh}]\alpha_h,$$

(3.2) $$\phi_{\sigma a_h}\alpha_h + \phi_{\sigma y_i}\eta_i + \phi_{\sigma y_i'}\eta_i' = 0,$$

(3.3) $$\psi_{\mu a_h}\alpha_h + \psi_{\mu y_i}\eta_i = 0,$$

(3.4) $$c_{\rho h}\alpha_h + \int_{x_1}^{x_2} (f_{\rho a_h}\alpha_h + f_{\rho y_i}\eta_i + f_{\rho y_i'}\eta_i')dx = 0,$$

in which $c_{\rho h}$ is defined as

$$c_{\rho h} = g_{\rho a_h} + f_\rho x_{sh} \Big|_{s=1}^{s=2},$$

and $x_{sh} = x_{sa_h}$, $y_{ish} = y_{isa_h}$, there exist constants $l_0 \geqq 0$, l_ρ, $l_{\mu s}$, continuously differentiable functions $m_\sigma(x)$, and continuous functions $\lambda_\mu(x)$ which do not all vanish simultaneously at any point x, such that if

(3.5) $$F(a, x, y, y', l, m, \lambda) \equiv l_0 f + l_\rho f_\rho + m_\sigma(x)\phi_\sigma + \lambda_\mu(x)\psi_\mu,$$

(3.6) $$G(a, l) \equiv l_0 g + l_\rho g_\rho + l_{\mu s}g_{\mu s},$$

then the following conditions, in addition to (1.2), (1.3), (1.4) and (1.5), hold along C:

(3.7) $$\frac{d}{dx} F_{y_i'} = F_{y_i};$$

(3.8) $$G_{a_h} + \left[(F - y_i'F_{y_i'})x_{sh} + F_{y_i'}y_{ish} \right]_{s=1}^{s=2} + \int_{x_1}^{x_2} F_{a_h}dx = 0;$$

$$E(a, x, y, y', Y', l, m, \lambda)$$

(3.9) $$\equiv F(a, x, y, Y', l, m, \lambda) - F(a, x, y, y', l, m, \lambda)$$
$$- (Y_i' - y_i')F_{y_i'}(a, x, y, y', l, m, \lambda) \geqq 0$$

whenever $(a, x, y, Y') \neq (a, x, y, y')$ and satisfies (1.3) and (2.1);

(3.10) $$F_{y_i'y_j'}(a, x, y, y', l, m, \lambda)\pi_i\pi_j \geqq 0$$

whenever $\pi \neq 0$ and is such that

(3.11) $\phi_{\sigma y_i'}(a, x, y, y')\pi_i = 0, \qquad \psi_{\mu y_i}(a, x, y)\pi_i = 0;$

(3.12) $J_2(\gamma) \equiv b_{hk}\alpha_h\alpha_k + \displaystyle\int_{x_1}^{x_2} 2\omega(\alpha, x, \eta, \eta')dx \geqq 0$

where

$$b_{hk} = G_{a_h a_k} + \left[(F - y_i' F_{y_i'})x_{shk} + F_{y_i'}y_{ishk} + F_{a_k}x_{sh} + F_{a_h}x_{sk} \right.$$
$$\left. + (F_x - y_i'F_{y_i})x_{sh}x_{sk} + F_{y_i}(x_{sh}y_{isk} + x_{sk}y_{ish}) \right]_{s=1}^{s=2},$$
$$2\omega = F_{y_iy_j}\eta_i\eta_j + 2F_{y_iy_j'}\eta_i\eta_j' + F_{y_i'y_j'}\eta_i'\eta_j' + 2F_{y_ia_h}\eta_i\alpha_h$$
$$+ 2F_{y_i'a_h}\eta_i'\alpha_h + F_{a_ha_k}\alpha_h\alpha_k.$$

In fact, it is known [2] that if C is of class D' and if γ satisfies (3.1), (3.2), (3.4) and

(3.13) $\phi_{\mu a_h}\alpha_h + \phi_{\mu y_i}\eta_i + \phi_{\mu y_i'}\eta_i' = 0,$

(3.14) $g_{\mu 1 a_h}\alpha_h = 0,$

then[3] there exist continuous multipliers $l_0 \geqq 0$, l_ρ, l_μ, $m_\sigma(x)$, $m_\mu(x)$, which do not all vanish simultaneously at any point x, such that if

(3.15) $F_0 = l_0 f + l_\rho f_\rho + m_\sigma \phi_\sigma - m_\mu \phi_\mu, \qquad G_0 = l_0 g + l_\rho g_\rho + l_\mu g_{\mu 1},$

then conditions (3.8), (3.9), (3.10) and (3.12) hold when F and G are replaced by F_0 and G_0. Moreover, we also have

(3.16) $F_{0y_i'} = \displaystyle\int_{x_1}^{x} F_{0y_i}dx + c_i$

for some constants c_i. An examination of the proof of this multiplier rule shows that if C is known to be of class C'', then the multipliers $m_\sigma(x)$, $m_\mu(x)$ are of class C'. Furthermore, (3.16) can be written in the form (3.7) with F replaced by F_0. If we now define $\lambda_\mu(x)$ and $l_{\mu s}$ by the equations

(3.17) $\lambda_\mu(x) = m_\mu'(x), \quad l_{\mu 1} = l_\mu + m_\mu(x_1), \quad l_{\mu 2} = -m_\mu(x_2),$

then the multipliers l_0, l_ρ, $l_{\mu s}$, $m_\sigma(x)$, $\lambda_\mu(x)$ have the differentiability and continuity properties which they were asserted to have, and

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[3] Cox derives these conditions for the nonisoperimetric problem. But the derivation of the necessary conditions for the isoperimetric problem from those for the nonisoperimetric problem is well known.

they vanish simultaneously at some point x if and only if the multipliers l_0, l_ρ, l_μ, $m_\sigma(x)$, $m_\mu(x)$ vanish simultaneously at some point x. If F and G are defined by (3.5) and (3.6) with these values of the multipliers, it is easy to see that conditions (3.7), (3.8), (3.9), (3.10) and (3.12) hold as stated when they hold for F_0 and G_0. Finally we observe that when C satisfies (1.2) and (1.5) and γ satisfies (3.1), then γ satisfies (3.13) and (3.14) if and only if γ satisfies (3.3).

4. **The sufficiency theorems.** These are now immediate consequences of the sufficiency theorems for the isoperimetric problem of Bolza [5, 7]. If for every nonidentically vanishing variation which satisfies (3.1), (3.2), (3.3) and (3.4) there exist multipliers (l, m, λ) satisfying the differentiability and continuity properties mentioned in §3 such that equations (3.7) and (3.8) hold, and for which

$$(4.1) \qquad F_{y_i' y_j'} \pi_i \pi_j > 0, \qquad J_2(\gamma) > 0$$

whenever $\pi \neq 0$ and is such that equations (3.11) hold, then [7] C gives $I(C)$ a proper weak relative minimum in the class of arcs satisfying (1.2), (1.3), (1.4) and (1.5). If in addition to (4.1) we require that (3.9) hold whenever the point $(a, x, y, y', l, m, \lambda)$ is in a neighborhood of those belonging to C (l_0 being held fast) and (a, x, y, Y') satisfies (1.3) and (2.1), then [5] C gives $I(C)$ a proper strong relative minimum in the class of arcs satisfying (1.2), (1.3), (1.4) and (1.5).

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