ON A GENERALIZATION OF THE STIELTJES MOMENT PROBLEM

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The "generalised moment problem"

$$(1) \qquad \int_0^\infty t^{\lambda_n} d\alpha(t) = \mu_n \qquad (0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots < \lambda_n \to \infty)$$

is said to be determined if there is at most one increasing function $\alpha(t)$ satisfying (1) and normalized by $\alpha(0)=0$. R. P. Boas, Jr., who first considered this problem [1][1] gave conditions under which (1) is determined. These do not include the best known result in the classical case $\lambda_n = n$, namely Carleman's criterion: If $\lambda_n = n$ and $\sum \mu_n^{-1/2n} = \infty$, then (1) is determined. I shall now prove a theorem including Carleman's test as a special case. On the other hand this theorem will not include the results of Boas, as I shall from now on assume

$$(2) \qquad \lambda_{n+1} - \lambda_n > c \qquad (n = 1, 2, \cdots)$$

for some $c > 0$.

Let

$$\psi(r) = \exp \left\{ \sum_{0<\lambda_\nu \leq r} \lambda_\nu^{-1} \right\}.$$

THEOREM. *If there are a non-increasing function $\phi(r)$ and positive constants A and a such that*

$$\psi(r) > A(r/\phi(r))^a$$

and if

$$(3) \qquad \sum_2^\infty \frac{\lambda_n - \lambda_{n-1}}{\mu_n^{1/a\lambda_n} \phi(\lambda_{n-1})} = \infty,$$

then (1) *is determined.*

The proof is based on the following lemma.

LEMMA. *If* (2) *is the case, then*

$$G(z) = \prod_{\nu=1}^\infty \frac{\lambda_\nu + z}{\lambda_\nu - z} e^{-2z/\lambda}$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

is regular apart from poles at the λ_ν *and for some constant* B

$$|G(z)| < B^x(\psi(r))^{-x} \qquad (z = x + iy = re^{i\theta})$$

in $x \geqq 0$ *except in circles of radius* $c/3$ *round the* λ_ν.

This lemma is proved in [2].

PROOF OF THE THEOREM. We must prove that two increasing solutions, $\alpha_1(t)$ and $\alpha_2(t)$, of (1) can differ by a constant only. Consider

$$F(z) = \frac{1}{2}\int_0^\infty t^z d(\alpha_1 - \alpha_2).$$

$F(z)$ is regular in $\Re z = x > 0$ and

$$|F(z)| < \frac{1}{2}\int_0^\infty t^x d(\alpha_1 + \alpha_2) \leqq (v(x))^{ax},$$

say. Since $(\int_0^\infty t^x d(\alpha_1+\alpha_2)/\int_0^\infty d(\alpha_1+\alpha_2))^{1/x}$ is an increasing function of x, by Hölder's inequality, we may choose

$$v(x) = K\mu_n^{1/a\lambda_n} \qquad (\lambda_{n-1} < x \leqq \lambda_n). \tag{4}$$

Also $F(\lambda_n) = 0$ $(n = 1, 2, \cdots)$, but unless $\alpha_1(t) - \alpha_2(t) =$ const., $F(z)$ does not vanish identically. It is therefore sufficient to prove that $F(z)$ is identically zero.

If $G(z)$ is the function defined in the lemma, let

$$H(z)s^{-z} = F(z/a)G(z/a)z^z e^{-C(1+z)}(1 + z)^{-2}s^{-z}.$$

This function is regular in $\Re z > 0$. Also, if $z = x + iy = re^{i\theta}$

$$\begin{aligned} |F(z/a)G(z/a)| &\leqq (v(x/a)\phi(r/a)BA^{-1}ar^{-1})^x \\ &\leqq (v(x/a)\phi(x/a)BA^{-1}ar^{-1})^x, \end{aligned}$$

$$|z^z| = r^x e^{-r\theta\sin\theta} \leqq r^x e^{-\pi|y|/2+x},$$

since $\theta\sin\theta \geqq \pi|\sin\theta|/2 - \cos\theta$ for $|\theta| \leqq \pi/2$;

$$|s^{-z}| = |s|^{-x}e^{y\arg s}.$$

Therefore

$$|H(z)s^{-z}| < (v(x/a)\phi(x/a)|s|^{-1})^x e^{-(\pi/2-|\arg s|)|y|}(1 + r)^{-2}, \tag{5}$$

provided that C is taken sufficiently large. Consider now

$$g(s) = \int_{1-i\infty}^{1+i\infty} H(z)s^{-z}dz. \tag{6}$$

Because of (5) the integral is uniformly convergent for $|s| \geqq 1$, $|\arg s| \leqq \pi/2$. In particular $g(s)$ is a regular function of s in $|s| > 1$, $|\arg s| < \pi/2$. It also follows from (5) that the line of integration in (6) may be shifted to any other line $x = b > 0$. Taking $b = \xi$ and using (5) gives

$$(7) \qquad |g(s)| < 2(v(\xi/a)\phi(\xi/a))^{\xi} |s|^{-\xi} \qquad (|\arg s| \leqq \pi/2)$$

for every $\xi > 0$.

By a theorem due to Carleman and Ostrowski (7) implies that $g(s)$ vanishes identically, if

$$(8) \qquad \int_1^{\infty} (v(\xi/a)\phi(\xi/a))^{-1} d\xi = \infty$$

(see [3], in particular Satz IV and §14).

By (4)

$$\int_{a\lambda_{n-1}}^{a\lambda_n} (v(\xi/a)\phi(\xi/a))^{-1} d\xi \geqq a \frac{\lambda_n - \lambda_{n-1}}{\mu_n^{1/a\lambda_n}\phi(\lambda_{n-1})},$$

so that (3) implies (8). Therefore $g(s)$ vanishes identically. By a well known uniqueness theorem for the Mellin transform this implies that $H(z)$ is zero and so $F(z)$ must be equal to zero everywhere. Q.e.d.

REFERENCES

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