ON THE SINGULARITIES OF A CLASS OF FUNCTIONS ON THE UNIT CIRCLE

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Pólya has suggested and Szegö and others have proved the following theorem.[1]

THEOREM. *Let $f(z)$ be a function regular in the whole plane including $z=\infty$ except at $z=1$. Let*

$$f(z) = \begin{cases} \sum a_n z^n, & |z| < 1, \\ \sum b_n/z^n, & |z| > 1. \end{cases}$$

If $a_n=O(n^k)$ and $b_n=O(n^k)$ then $f(z)$ is a rational function.

The above theorem is generalized in this paper as follows.

THEOREM 1. *Let $f(z)$ be regular in the whole plane including $z=\infty$, except possibly at a certain set S of points on $|z|=1$ (the set S being not everywhere dense on the complete circumference of the unit circle). Let*

$$f(z) = \begin{cases} \sum a_n z^n, & |z| < 1, \\ \sum b_n/z^n, & |z| > 1, \end{cases}$$

and let $a_n=O(n^k)$, $b_n=O(n^k)$; then the following results hold.

(i) *Every isolated singularity on $|z|=1$ will be a pole of order not exceeding $k+1$.*

(ii) *If there are only a finite number of singularities on $|z|=1$, then $f(z)$ is a rational function.*

THEOREM 2. *There exists a function satisfying the hypothesis of Theorem 1 and having an infinite number of singularities on the unit circle; also there exists a function satisfying the same hypothesis and having no isolated singularities.*

LEMMA 1. *Let $f(z)$ be an integral function and let*

$$I_p(r) = \int_0^{2\pi} |f(re^{i\phi})|^p d\phi \qquad \text{where } p > 0$$

be bounded on a sequence of circles $r=r_n$ tending to infinity, for some $p>0$. Then $f(z)$ reduces to a constant.

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[1] J. Deutschen Math. Verein vol. 40 (1931) Aufgaben und Lösungen p. 81 (Polyá); ibid. vol. 43 (1934) Aufgaben und Lösungen pp. 13–16 (Szegö and others).

PROOF. $|f(z)|^p$ is subharmonic in any region of the z-plane. By Poisson's integral formula

$$|f(z)|^p \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,|f(Re^{i\phi})|^p d\phi}{R^2 + r^2 - 2rR\cos(\phi - \theta)}$$

where $|z| = r < R = r_n$. Hence

$$|f(z)|^p \leqq \frac{1}{2\pi}\,\frac{R^2 - r^2}{(R - r)^2}\int_0^{2\pi} |f(Re^{i\phi})|^p d\phi \leqq K\,\frac{R + r}{R - r}\,.$$

Putting $R = 2r$ we get

$$|f(z)|^p \leqq 3K/2 \quad \text{on} \quad |z| = R/2.$$

Hence $f(z)$ is bounded on $|z| = r_n/2$ and so it reduces to a constant.

LEMMA 2. *Let $f(z)$ be regular for $|z| \geqq H$ except probably at infinity; and let*

$$\int_0^{2\pi} |f(z)|^p d\phi \qquad (p > 0)$$

be bounded on a sequence of circles $|z|r = r_n$ tending to infinity. Then $f(z)$ is regular at infinity.

PROOF. We can write $f(z) = g(z) + h(z)$ where $g(z)$ is an integral function and $h(z)$ regular at infinity. Since $h(z)$ is bounded at infinity, it follows from Minkowski's inequality when $p > 1$ (and still simpler when $p \leqq 1$) that $\int_0^{2\pi}|g(z)|^p d\phi$ $(z = re^{i\phi})$ is bounded on a sequence of circles $r = r_n$ tending to infinity. Hence $g(z)$ is constant by Lemma 1 and so $f(z)$ is regular at infinity.

PROOF OF THEOREM 1. To prove that every isolated singularity will be a pole, it is enough to prove that if $z = 1$ is an isolated singularity, it is a pole since every other singularity can be brought to $z = 1$ by a rotation. We suppose that k is a positive integer. We have already supposed that $z = 1$ is an isolated singularity of $f(z)$. Let $x = (1 + z)/(1 - z)$. This transforms the unit circle in the z-plane into the imaginary axis in the x-plane and $z = 1$ corresponds to $x = \infty$. The function $\phi(x) = f(z)$ given by the above relation is therefore regular for $|x| \geqq R_0$ (where R_0 is some number), except at $x = \infty$.

From our assumptions about the coefficients we obtain

$$|f(z)| \leqq \frac{c}{|1 - |z||^{k+1}}$$

in the neighbourhood of the circle $|z| = 1$. Hence

$$|(1-z)^{k+1}f(z)| \leqq c\left|\frac{1-z}{1-|z|}\right|^{k+1}.$$

Let $\psi(x)=(1-z)^{k+1}f(z)$ where $z=(x-1)/(x+1)$. Then

$$|\psi(x)| \leqq \frac{2^{k+1}}{|x+1|^{k+1}} \frac{c}{|1-|(x-1)/(x+1)||^{k+1}}$$

$$\leqq \frac{c_1}{||x+1|-|x-1||^{k+1}}.$$

Let $x=\rho e^{i\gamma}$. Then $|x+1|^2-|x-1|^2=4\rho\cos\gamma$. Hence

$$|\psi(x)| \leqq \frac{c_1\{|x+1|+|x-1|\}^{k+1}}{\{||x+1|^2-|x-1|^2|\}^{k+1}}$$

$$\leqq c_1\left\{\frac{|x+1|+|x-1|}{4\rho|\cos\gamma|}\right\}^{k+1}$$

$$\leqq \frac{c_2}{|\cos\gamma|^{k+1}}$$

if $\rho \geqq \rho_0$ is sufficiently large. Hence

$$|\psi(x)|^{1/2(k+1)} \leqq \frac{c_3}{|\cos\gamma|^{1/2}}$$

except when $\gamma=\pm\pi/2$. Hence

$$\int_0^{2\pi}|\psi(x)|^{1/2(k+1)}d\gamma \leqq \int_0^{2\pi}\frac{c_3 d\gamma}{(|\cos\gamma|)^{1/2}}$$

which is a convergent integral. Hence

$$\int_0^{2\pi}|\psi(x)|^{1/2(k+1)}d\gamma$$

is bounded and so, by Lemma 2, $\psi(x)$ is regular at infinity. Hence

$$(1-z)^{k+1}f(z)$$

is regular at $z=1$, which shows that $f(z)$ has a pole of order not exceeding $k+1$ at $z=1$.

This proves (i). To prove (ii) it follows by part (i) that each of the finite number of singularities on $|z|=1$ is a pole. Hence $f(z)$ is a regular function throughout the z-plane including infinity, except for a finite number of poles. Hence $f(z)$ is a rational function.

PROOF OF THEOREM 2. Consider the function

$$f(z) = \sum_{1}^{\infty} \frac{1}{2^n} \frac{1}{(z - \alpha_n)}$$

where (α_n) is any sequence of points on $|z| = 1$. *If* the sequence (α_n) has only one limit point the above function $f(z)$ has a pole at each of these points α_n and an essential singularity at the limit point of the sequence (α_n). It is regular elsewhere. If

$$f(z) = \sum_{0}^{\infty} a_p z^p \qquad \text{for } |z| < 1$$

then

$$a_p = - \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{\alpha_n^{p+1}}$$

and therefore $|a_p| \leqq 1$. Similarly if $f(z) = \sum b_p / z^p$ $(|z| > 1)$ then $|l_p| \leqq 1$. Hence a_p and b_p are certainly $O(n^k)$ for any $k \geqq 0$. To prove the second part, it is enough to take (α_n) in the above example to be everywhere dense on some arc of the unit circle, the arc not being the whole of the circumference. The function $f(z)$ will have a non-isolated essential singularity at every point of this arc and the coefficients a_n and b_n are bounded.

This example shows that the part (ii) of Theorem 1 is in a sense the best possible result, for the function constructed satisfies the conditions on the coefficients while it is not a rational function since it has an infinite number of singularities on the unit circle.

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