THE STRONG SUMMABILITY OF DOUBLE FOURIER SERIES

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1. **Introduction.** Corresponding to the well known theorem of Fejér-Lebesgue, we have for the double Fourier series the following proposition:

If $f \log^+|f|$ is Lebesgue integrable on the square $(-\pi \leqq x \leqq \pi, -\pi \leqq y \leqq \pi)$, then the Fejér mean $\sigma_{m,n}(x, y)$ of $f(x, y)$ tends to $f(x, y)$ almost everywhere as m and n independently increase indefinitely. Moreover, for every increasing function $\phi(t)$ satisfying the conditions

$$\phi(0) = 0, \qquad \liminf_{t\to\infty} \frac{\phi(t)}{t \log t} = 0,$$

there is a function $f(x, y)$ such that $\phi(|f|)$ is integrable and that $\sigma_{m,n}(x, y)$ does not converge almost everywhere.[1]

The latter half of this theorem shows that the analogue, in double Fourier series, of the Fejér-Lebesgue theorem is not a trivial extension of that of a function of a single variable.

The purpose of the present note is to discuss the strong summability[2] of double Fourier series. A double series $\sum a_{mn}$ is said to be strongly summable with the positive index k if there exists a constant s such that the expression

$$(1.1) \qquad \frac{1}{(m+1)(n+1)} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | s_{\mu,\nu} - s |^k$$

has the double limit zero as m and n increase without limit, where

$$s_{m,n} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}.$$

It is easily seen from Hölder's equality that the summability says more for larger k.

Suppose now that $f(x, y)$ is integrable in the Lebesgue sense over

Received by the editors May 16, 1945.

[1] B. Jessen, J. Marcinkiewicz and A. Zygmund [5]. The first example of a function $f(x, y) \in L$ with Fejér mean divergent everywhere was given by A. Zygmund; see S. Saks [8]. Numbers in brackets refer to the Bibliography at the end of the paper.

[2] A notion first introduced in Fourier series by G. H. Hardy and J. E. Littlewood [1]. For subsequent researches, see Hardy and Littlewood [2, 3], J. Marcinkiewicz [6] and A. Zygmund [12].

the square $Q\ (-\pi,\ -\pi;\ \pi,\ \pi)$ and is doubly periodic with period 2π in each variable. The Fourier series of $f(x, y)$ is

$$\sum_{m,n=0}^{\infty} \lambda_{m,n}[a_{m,n} \cos mx \cos ny + b_{m,n} \sin mx \cos ny + c_{m,n} \cos mx \sin ny + d_{m,n} \sin mx \sin ny], \tag{1.2}$$

where

$$\lambda_{m,n} = \begin{cases} 1/4 & \text{for } m = n = 0; \\ 1/2 & \text{for } m = 0, n > 0 \quad \text{or} \quad m > 0, n = 0; \\ 0 & \text{for } m > 0, n > 0; \end{cases}$$

and

$$a_{m,n} = \frac{1}{\pi^2} \int_Q \int f(x, y) \cos mx \cos ny dx dy,$$

and so on.

On writing

$$4\phi(u, v) \equiv \phi_{x,y}(u, v) = f(x + u, y + v) + f(x + u, y - v) + f(x - u, y + v) + f(x - u, y - v) - 4s,$$

and

$$\Phi_{x,y}^{(p)}(u, v) = \int_0^u \int_0^v |\phi(\xi, \eta)|^p d\xi d\eta \qquad (p \geqq 1)$$

the theorems obtained in this paper are as follows:

THEOREM I. *If $f(x, y) \in L^p$, $p > 1$, then the double Fourier series* (1.2) *is strongly summable to s for every positive index k whenever*[3]

$$\Phi_{x,y}^{(p)}(u, v) = o(uv). \tag{1.3}$$

THEOREM II. *If $f(x, y) \in L^p$, $p > 1$, then the Fourier series of $f(x, y)$ is strongly summable almost everywhere to $f(x, y)$ for every positive index k.*

The question whether the hypothesis in Theorem II may be replaced by $f \log^+ |f| \in L$ is unsettled in this note. Corresponding questions in Fourier series of a single variable have been answered affirmatively by Marcinkiewicz [6] and Zygmund [12]. Indeed, the theorem holds under the weaker hypothesis $f \in L$. We content ourselves with establishing the following theorem.

[3] We use the symbol $o(uv)$ to denote a function of u and v such that $\lim_{u,v\to 0} o(uv)/uv = 0$.

THEOREM III. *If* $f(x, y) \log^+ |f(x, y)| \in L$ *and* $\sigma_{m,n}(x, y)$ *denotes the* (m, n)*th Fejér sum of the Fourier series of* $f(x, y)$, *then the relation*

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} |\sigma_{\mu,\nu}(x, y) - f(x, y)|^k = 0$$

holds true almost everywhere, where $k>0$.

2. **Lemmas.** Before proving our theorems, we prove a number of lemmas:

LEMMA 1. *If* $f(x, y) \in L^p$, $p>1$, *then*

$$\lim_{h,k\to 0} \frac{1}{hk} \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(x, y)dxdy = f(x_0, y_0)$$

at almost every point (x_0, y_0).

This theorem is due to Zygmund [11]. Compare also [5] and [9].

LEMMA 2. *If* $f(x, y) \in L^p$, $p>1$, *then at almost every point* (x, y),

$$\int_0^h \int_0^k |f(x \pm u, y \pm v) - f(x, y)|^p dudv = o(hk)$$

as $h, k \to 0$.

PROOF. Let α be a rational number, and E_α the set of points (x, y) such that

$$\frac{1}{hk} \int_0^h \int_0^k |f(x \pm u, y \pm v) - \alpha|^p dudv$$

does not tend to $|f(x, y) - \alpha|^p$ as $h, k \to 0$. In virtue of Lemma 1, E_α is of measure zero, and so also is the sum E of all E_α. Let (x, y) be not a point of E and let β be a rational number, then, by Minkowski's inequality,

$$\begin{aligned} \left\{\frac{1}{hk} \int_0^h \int_0^k |f(x \pm u, y \pm v) - f(x, y)|^p dudv\right\}^{1/p} \\ \leqq \left\{\frac{1}{hk} \int_0^h \int_0^k |f(x \pm u, y \pm v) - \beta|^p dudv\right\}^{1/p} \\ + \left\{\frac{1}{hk} \int_0^h \int_0^k |\beta - f(x, y)|^p dudv\right\}^{1/p}, \end{aligned}$$

which tends to $2|f(x, y) - \beta|$ as $h, k \to 0$. As $\beta \to f(x, y)$, the result follows.

LEMMA 3. *Let* $f(x, y) \in L^p$, $1 < p \leqq 2$, $1/p + 1/q = 1$, *and let the Fourier series of* $f(x, y)$ *be given in the complex form*:

$$f(x, y) \sim \sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} c_{\mu,\nu} e^{i(\mu x + \nu y)},$$

then

$$\left\{ \sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} | c_{\mu,\nu} |^q \right\}^{1/q} \leqq \left\{ \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} | f(x, y) |^p dx dy \right\}^{1/p}. \tag{2.1}$$

This is a double series analogue of the Young-Hausdorff theorem, and may be proved by the method of M. Riesz[4] with an obvious modification.

We also require the following formula of integration by parts:

$$\begin{aligned} \int_{a_1}^{a_2} du \int_{b_1}^{b_2} \rho \psi'(u) \psi''(v) du dv & \\ = \rho_1(a_2, b_2) \psi(a_2, b_2) &- \int_{a_1}^{a_2} \rho_1(u, b_2) \psi_u(u, b_2) du \\ &- \int_{b_1}^{b_2} \rho_1(a_2, v) \psi_v(a_2, v) dv + \int_{a_1}^{a_2} du \int_{b_1}^{b_2} \rho_1 \psi_{uv} dv, \end{aligned} \tag{2.2}$$

where

$$\psi(u, v) = \psi'(u) \psi''(v), \qquad \rho_1(u, v) = \int_{a_1}^{u} d\sigma \int_{b_1}^{v} \rho(\sigma, t) dt.$$

This formula is valid if ρ is integrable on $(a_1, b_1; a_2, b_2)$, ψ' is absolutely continuous on (a_1, a_2), and ψ'' is absolutely continuous on (b_1, b_2).

3. **Proof of Theorem I.** Without loss of generality, we may assume that $x = 0$, $y = 0$. So that

$$s_{m,n} = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(u, v) \frac{\sin (m + 1/2) u}{\sin u/2} \frac{\sin (n + 1/2) v}{\sin v/2} du dv.$$

We have to deduce

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | s_{\mu,\nu} - s |^k = o(uv) \tag{3.1}$$

from (1.3).

Write

[4] M. Riesz [7], see also A. Zygmund [13].

$$\pi^2(s_{\mu,\nu} - s) = \int_0^\pi \int_0^\pi \phi(u, v) \frac{\sin (m + 1/2)u}{\sin u/2} \frac{\sin (n + 1/2)v}{\sin v/2} dudv$$

$$= \int_0^\pi \int_0^\pi \phi(u, v) \left(\sin \mu u \cot \frac{u}{2} \sin \nu v + \sin \mu u \cot \frac{u}{2} \cos \nu v \right.$$

$$\left. + \cos \mu u \sin \nu v \cot \frac{v}{2} + \cos \mu u \cos \nu v \right) dudv$$

$$= I_1(\mu, \nu) + I_2(\mu, \nu) + I_3(\mu, \nu) + I_4(\mu, \nu),$$

and for $0 < \mu \leqq m$, $0 < \nu \leqq n$,

$$I_i(\mu, \nu) = \int_0^{m^{-1}} \int_0^{n^{-1}} + \int_0^{m^{-1}} \int_{n^{-1}}^{\pi} + \int_{m^{-1}}^{\pi} \int_0^{n^{-1}} + \int_{m^{-1}}^{\pi} \int_{n^{-1}}^{\pi}$$

$$= I_{i1}(\mu, \nu; m, n) + I_{i2}(\mu, \nu; m, n)$$

$$+ I_{i3}(\mu, \nu; m, n) + I_{i4}(\mu, \nu; m, n),$$

where $i = 1, 2, 3$. For brevity, we also write $I_{ij}(\mu, \nu)$ for $I_{ij}(\mu, \nu; m, n)$. Accordingly,

$$\pi^2(s_{\mu,\nu} - s) = \sum_{i=1}^{3} \sum_{j=1}^{4} I_{ij}(\mu, \nu) + I_4(\mu, \nu).$$

It follows from Minkowski's inequality that

$$\pi^2 \left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | s_{\mu,\nu} - s |^k \right\}^{1/k} \leqq \sum_{i=1}^{3} \sum_{j=1}^{4} \left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_{ij}(\mu, \nu) |^k \right\}^{1/k} + \left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_4(\mu, \nu) |^k \right\}^{1/k}. \tag{3.2}$$

In the first place, by the analogue of the Riemann-Lebesgue theorem[5] $I_4(\mu, \nu)$ tends to zero as $\mu, \nu \to \infty$. Hence

$$\left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_4(\mu, \nu) |^k \right\}^{1/k} = o(mn)^{1/k}. \tag{3.3}$$

Secondly, let us consider the integrals I_{11}, I_{21} and I_{31}. Write

$$K(u, v) \equiv K(u, v; \mu, \nu) = \sin \mu u \cot u/2 \sin \nu v \cot v/2,$$

then for $0 < u \leqq \pi$ and $0 < v \leqq \pi$ there is a constant A such that

$$uv \max (| K |, \mu^{-1} | K_u |, \nu^{-1} | K_v |, \mu^{-1}\nu^{-1} | K_{uv} |) \leqq A. \tag{3.4}$$

We also write

[5] W. H. Young [10, p. 138].

$$\Phi(u, v) = \int_0^u \int_0^v \phi(\xi, \eta) d\xi d\eta,$$

which is $o(uv)$ by (1.3). Then on applying (2.2),

$$
\begin{aligned}
I_{11}(\mu, \nu) &= \int_0^{m^{-1}} \int_0^{n^{-1}} \phi(u, v) K(u, v; \mu, \nu) du dv \\
&= \Phi(m^{-1}, n^{-1}) K(m^{-1}, n^{-1}) \\
&\quad - \int_0^{m^{-1}} \Phi(u, n^{-1}) K_u(u, n^{-1}) du \\
&\quad - \int_0^{n^{-1}} \Phi(m^{-1}, v) K_v(m^{-1}, v) dv \\
&\quad + \int_0^{m^{-1}} du \int_0^{n^{-1}} \Phi(u, v) K_{uv} dv.
\end{aligned}
\tag{3.5}
$$

Since $0 < \mu \leqq m$, $0 < \nu \leqq n$, it is easily seen from (3.4) and (3.5) that

$$I_{11}(\mu, \nu) = o(1). \tag{3.6}$$

In a similar manner, we can prove $I_{21}(\mu, \nu) = o(1)$, $I_{31}(\mu, \nu) = o(1)$. Hence we obtain

$$\left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| I_{i1}(\mu, \nu) \right|^k \right\}^{1/k} = o(mn)^{1/k} \qquad (i = 1, 2, 3). \tag{3.7}$$

Thirdly, we consider the integrals I_{14}, I_{24} and I_{34}. We have

$$
\begin{aligned}
I_{14}(\mu, \nu) &= \int_{m^{-1}}^{\pi} \int_{n^{-1}}^{\pi} \phi(u, v) \sin \mu u \cot \frac{u}{2} \sin \nu v \cot \frac{v}{2} du dv \\
&= \int_{m^{-1}}^{\pi} \sin \mu u \cot \frac{u}{2} du \\
&\quad \cdot \int_{n^{-1}}^{\pi} \cot \frac{v}{2} \left(\frac{\partial}{\partial v} \int_0^v \sin \nu y \phi(u, y) dy \right) dv \\
&= \int_{m^{-1}}^{\pi} \sin \mu u \cot \frac{u}{2} du \left(- \cot \frac{1}{2n} \int_0^{n^{-1}} \sin \nu y \phi(u, y) dy \right. \\
&\quad \left. + \frac{1}{2} \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} dv \int_0^v \sin \nu y \phi(u, y) dy \right) \\
&= I'_{14} + I''_{14},
\end{aligned}
\tag{3.8}
$$

say, where I'_{14} is equal to

$$- \cot \frac{1}{2n} \int_{m^{-1}}^{\pi} \cot \frac{u}{2} \left(\frac{d}{du} \int_0^u \int_0^{n^{-1}} \sin \mu x \sin \nu y \phi(x, y) dx dy \right) du$$

$$= \cot \frac{1}{2m} \cot \frac{1}{2n} \int_0^{m^{-1}} \int_0^{n^{-1}} \sin \mu x \sin \nu y \phi(x, y) dx dy$$

$$- \frac{1}{2} \cot \frac{1}{2n} \int_{m^{-1}}^{\pi} \csc^2 \frac{u}{2}\, du \int_0^u \int_0^{n^{-1}} \sin \mu x \sin \nu y \phi(x, y) dx dy.$$

Let $c_{\mu,\nu}(\alpha, \beta)$ denote the (μ, ν)th Fourier coefficient of the odd-odd function $\chi(x, y)$ which is equal to $\phi(x, y)$ in the rectangle $(0, \alpha; 0, \beta)$ and to zero elsewhere. Then we may write

$$I'_{14} = \frac{\pi^2}{4} \cot \frac{1}{2m} \cot \frac{1}{2n} c_{\mu,\nu}\left(\frac{1}{m}, \frac{1}{n}\right) - \frac{\pi^2}{8} \cot \frac{1}{2n} \int_{m^{-1}}^{\pi} \csc^2 \frac{u}{2} c_{\mu,\nu}\left(u, \frac{1}{n}\right) du, \tag{3.9}$$

and I''_{14} may be written as

$$\begin{aligned} &\frac{1}{2} \int_{m^{-1}}^{\pi} \cot \frac{u}{2}\, du \\ &\quad \cdot \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} \left(\frac{\partial}{\partial u} \int_0^u \int_0^v \sin \mu x \sin \nu y \phi(x, y) dx dy \right) dv \\ &= \frac{\pi^2}{8} \int_{m^{-1}}^{\pi} \cot \frac{u}{2} \left(\frac{\partial}{\partial u} \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} c_{\mu,\nu}(u, v) dv \right) du \\ &= -\frac{\pi^2}{8} \cot \frac{1}{2m} \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} c_{\mu,\nu}\left(\frac{1}{m}, v\right) dv \\ &\quad + \frac{\pi^2}{16} \int_{m^{-1}}^{\pi} \int_{n^{-1}}^{\pi} \csc^2 \frac{u}{2} \csc^2 \frac{v}{2} c_{\mu,\nu}(u, v) du dv. \end{aligned} \tag{3.10}$$

It follows from (3.8), (3.9) and (3.10) that

$$\begin{aligned} &\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_{14}(\mu, \nu) |^k \right)^{1/k} \\ &\leqq A \cot \frac{1}{2m} \cot \frac{1}{2n} \left(\sum_{\mu=\nu}^{m} \sum_{\nu=0}^{n} \left| c_{\mu,\nu}\left(\frac{1}{m}, \frac{1}{n}\right) \right|^k \right)^{1/k} \\ &\quad + A \cot \frac{1}{2n} \int_{m^{-1}}^{\pi} \csc^2 \frac{u}{2} \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| c_{\mu,\nu}\left(u, \frac{1}{n}\right) \right|^k \right)^{1/k} du \end{aligned}$$

$$+ A \cot \frac{1}{2m} \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| c_{\mu,\nu}\left(\frac{1}{m}, v\right) \right|^k \right)^{1/k} dv$$

$$+ A \cot \frac{1}{2m} \cot \frac{1}{2n} \int_{m^{-1}}^{\pi} \int_{n^{-1}}^{\pi} \csc^2 \frac{u}{2} \csc^2 \frac{v}{2}$$

$$\cdot \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | c_{\mu,\nu}(u, v) |^k \right)^{1/k} dudv.$$

Now we assume, without loss of generality, that $k>2$, $k'=k/(k-1)<p$, so that by Lemma 3,

$$\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | c_{\mu,\nu}(u, v) |^k \right)^{1/k} \leqq \left(\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} | \chi(x, y) |^{k'} dxdy \right)^{1/k'}$$

$$= \left(\frac{1}{4\pi^2} \int_{-u}^{u} \int_{-v}^{v} | \phi(x, y) |^{k'} dxdy \right)^{1/k'}$$

$$= o(uv)^{1/k'},$$

since the condition (1.3) is satisfied a fortiori when p is replaced by the smaller index k'. Therefore

$$\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_{14}(\mu, \nu) |^k \right)^{1/k} \leqq Amno(mn)^{-1/k'} + An \int_{m^{-1}}^{\pi} \left(\frac{u}{n} \right)^{1/k'} \frac{du}{u^2}$$

$$+ Am \int_{n^{-1}}^{\pi} \left(\frac{v}{m} \right)^{1/k'} \frac{dv}{v^2}$$

$$+ A \int_{m^{-1}}^{\pi} \int_{n^{-1}}^{\pi} \frac{(uv)^{1/k'}}{u^2 v^2} dudv = o(mn)^{1/k}.$$

The integral $I_{24}(\mu, \nu)$ is equal to

$$\int_{m^{-1}}^{\pi} \cot \frac{u}{2} \left(\frac{d}{du} \int_0^u dx \int_{n^{-1}}^{\pi} \phi(x, y) \sin \mu x \cos \nu y dy \right) du$$

$$= \frac{\pi^2}{4} \cot \frac{1}{2m} \left[c'_{\mu,\nu}\left(\frac{1}{m}, \pi\right) - c'_{\mu,\nu}\left(\frac{1}{m}, \frac{1}{n}\right) \right]$$

$$- \frac{\pi^2}{8} \int_{m^{-1}}^{\pi} \csc^2 \frac{u}{2} \left[c'_{\mu,\nu}(u, \pi) - c'_{\mu,\nu}\left(u, \frac{1}{n}\right) \right] du,$$

where the $c'_{\mu,\nu}(\alpha, \beta)$ $(\mu, \nu=0, 1, 2, \cdots)$ denote the Fourier coefficients of the odd-even function $\chi'(x, y)$ which is equal to $\phi(x, y)$ in the rectangle $(0, \alpha; 0, \beta)$ and to zero elsewhere. In virtue of Minkowski's inequality and Lemma 3, it is easily seen that

$$\left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\left|I_{24}(\mu,\nu)\right|^{k}\right)^{1/k}=o(mn)^{1/k}.$$

The integral I_{34} can be treated in the same manner as I_{14}. We omit the details. Collecting the above results, we obtain

$$\left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\left|I_{i4}(\mu,\nu)\right|^{k}\right)^{1/k}=o(mn)^{1/k}\qquad(i=1,2,3). \tag{3.11}$$

Fourthly, we estimate the integrals I_{12}, I_{22} and I_{32}. We have

$$\begin{aligned} I_{12}(\mu,\nu) &= \int_{0}^{m^{-1}}\int_{n^{-1}}^{\pi}\phi(u,v)\sin\mu u\cot\frac{u}{2}\sin\nu v\cot\frac{v}{2}\,dudv \\ &= \int_{0}^{m^{-1}}\cot\frac{u}{2}\sin\mu udu \\ &\quad\cdot\int_{n^{-1}}^{\pi}\cot\frac{v}{2}\left[\frac{\partial}{\partial v}\int_{0}^{v}\phi(u,y)\sin\nu ydy\right]dv \\ &= \int_{0}^{m^{-1}}\cot\frac{u}{2}\sin\mu u\left\{-\frac{\pi}{2}\cot\frac{1}{2n}c_{\nu}\left(u,\frac{1}{n}\right)\right. \\ &\quad\left.+\frac{\pi}{4}\int_{n^{-1}}^{\pi}\csc^{2}\frac{v}{2}\,c_{\nu}(u,v)dv\right\}du, \end{aligned} \tag{3.12}$$

where $c_{\nu}(\alpha,\beta)$ denotes the νth Fourier coefficient of the odd function $\psi(u,\beta)$ which is equal to $\phi(u,v)$ for $0\leqq v\leqq\beta$ and to zero for $\beta<v<\pi$. It follows from Young-Hausdorff's inequality that

$$\begin{aligned} \left(\sum_{\nu=1}^{n}\left|c_{\nu}(u,v)\right|^{k}\right)^{1/k} &\leqq \left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\psi(u,y)\right|^{k'}dy\right)^{1/k'} \\ &= \left(\frac{1}{\pi}\int_{-v}^{v}\left|\phi(u,y)\right|^{k'}dy\right)^{1/k'}, \end{aligned}$$

so that

$$\begin{aligned} &\left(\sum_{\nu=0}^{n}\left|I_{12}(\mu,\nu)\right|^{k}\right)^{1/k} \\ &\leqq A\cot\frac{1}{2n}\int_{0}^{m^{-1}}\mu\left(\int_{-n^{-1}}^{n^{-1}}\left|\phi(u,y)\right|^{k'}dy\right)^{1/k'}du \\ &\quad+A\int_{0}^{m^{-1}}\mu du\int_{n^{-1}}^{\pi}\csc^{2}\frac{v}{2}\left(\int_{-v}^{v}\left|\phi(u,y)\right|^{k'}dy\right)^{1/k'}dv. \end{aligned} \tag{3.13}$$

Hölder's inequality gives

$$\int_0^{m^{-1}}\left(\int_{-v}^{v}|\phi(u,y)|^{k'}dy\right)^{1/k'}du$$
$$\leqq m^{1/k'-1}\left(\int_0^{m^{-1}}\int_{-v}^{v}|\phi(u,y)|^{k'}dydu\right)^{1/k'} = m^{1/k'-1}o\left(\frac{v}{m}\right)^{1/k'}.$$

Hence (3.13) is reduced to

$$\left(\sum_{\nu=0}^{n}|I_{12}(\mu,\nu)|^k\right)^{1/k}$$
$$\leqq \mu\cot\frac{1}{2n}m^{1/k'-1}o(mn)^{-1/k'} + \mu m^{1/k'-1}\int_{n^{-1}}^{\pi}v^{-2}o\left(\frac{v}{m}\right)^{1/k'}dv$$
$$= o(n)^{1/k}.$$

Thus we obtain $(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}|I_{12}(\mu,\nu)|^k)^{1/k}=o(mn)^{1/k}$. The integrals I_{22} and I_{32} may be treated in a similar manner as above. The following relations are thus established:

$$\left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}|I_{i2}(\mu,\nu)|^k\right)^{1/k} = o(mn)^{1/k} \qquad (i=1,2,3). \tag{3.14}$$

Finally, we have to consider the integrals I_{13}, I_{23} and I_{33}. The discussion of I_{13} is the same as I_{12}, and the integral I_{23} has been treated implicitly in the discussion of I_{24}. It remains therefore only to deal with I_{33}. Regard the integrals

$$I_{33}(\mu,\nu) = \int_{m^{-1}}^{\pi}\cos\mu u du\int_0^{n^{-1}}\phi(u,v)\sin\nu v\cot\frac{v}{2}dv$$
$$(\mu = 0, 1, 2, \cdots)$$

as the Fourier coefficients of the function of u which is equal to

$$\int_0^{n^{-1}}\phi(u,v)\sin\nu v\cot\frac{v}{2}dv \qquad \text{for } m^{-1}\leqq u\leqq\pi,$$

and to zero for $-\pi\leqq u<m^{-1}$, then by Hausdorff's inequality,

$$\left\{\sum_{\mu=0}^{m}|I_{33}(\mu,\nu)|^k\right\}^{1/k}$$
$$\leqq\left(\frac{1}{4\pi^2}\int_{m^{-1}}^{\pi}\left|\int_0^{n^{-1}}\sin\nu v\cot\frac{v}{2}\phi(u,v)dv\right|^{k'}du\right)^{1/k} \tag{3.15}$$
$$\leqq A_\nu\left(\int_{m^{-1}}^{\pi}\left(\int_0^{n^{-1}}|\phi(u,v)|dv\right)^{k'}du\right)^{1/k'}.$$

It follows from Hölder's inequality that

$$\int_0^{n^{-1}} |\phi(u, v)| \, dv \leqq n^{1/k'-1}\left(\int_0^{n^{-1}} |\phi(u, v)|^{k'} dv\right)^{1/k'}$$

so that

$$vn^{1/k'-1}\left(\int_{m^{-1}}^{\pi} du \int_0^{n^{-1}} |\phi(u, v)|^{k'} dv\right)^{1/k'} = vn^{1/k'-1}O(n^{-1/k'}) = O(1).$$

Hence from (3.15) it results that $\{\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |I_{33}(\mu, \nu)|^k\}^{1/k} = o(mn)^{1/k}$. The following relations are thus proved:

$$\left\{\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |I_{i3}(\mu, \nu)|^k\right\}^{1/k} = o(mn)^{1/k} \qquad (i = 1, 2, 3). \tag{3.16}$$

Collecting the results (3.2), (3.3), (3.7), (3.11), (3.14), and (3.16) we obtain (3.1). Theorem I is thus proved.

4. **Proof of Theorem II.** On account of Theorem I, it suffices to show that the condition (1.3) is satisfied almost everywhere when $s = f(x, y)$. Observing

$$\begin{aligned} 4 |\phi_{x,y}(u, v)| \leqq & |f(x + u, y + v) - f(x, y)| \\ & + |f(x + u, y - v) - f(x, y)| \\ & + |f(x - u, y + v) - f(x, y)| \\ & + |f(x - u, y - v) - f(x, y)|, \end{aligned}$$

and employing Minkowski's inequality, we immediately obtain the desired result from Lemma 2.

5. **Proof of Theorem III.** The proof depends upon the following two lemmas:

LEMMA 4. *Theorem* III *holds good when* $f(x, y)$ *is bounded.*

Since a bounded function belongs to L^p, $p > 1$, the lemma follows from Theorem II.

LEMMA 5. *Let* $h(x)$ *be a function such that* $h \log^+ |h| \in L\ (-\pi, \pi)$. *Let* $\beta_m = \beta_m(x, h)$ $(m = 0, 1, 2, \cdots)$ *be the Fejér sums of the Fourier series of* $h(x)$, *and* $\beta^*(x) = \sup_m |\beta_m(x)|$, *then*

$$\int_{-\pi}^{\pi} \beta^*(x)dx \leqq A \int_{-\pi}^{\pi} |h| \log^+ |h| \, dx + B,$$

where A and B are absolute constants.

This lemma is due to Hardy and Littlewood [**4**]. See also [**13**, p. 248].

Before proving the theorem, we extend Lemma 5 to the case of two variables. Let, for fixed y,

$$g(x, y) = \sup_m \beta_m(x; |f|).$$

Integrating this equation with respect to y, we obtain

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x, y)dxdy \le 2\pi A \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| dxdy + 2\pi B. \tag{5.1}$$

Writing $K_n(x)$ for the Fejér kernel, we have

$$\sigma_{m,n}(x, y; f) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u, v)K_m(x - u)K_n(y - v)dudv.$$

It follows that

$$|\sigma_{m,n}(x, y; f)| \le \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(y - v)g(x, v)dv.$$

In virtue of Lebesgue's theorem, the last expression tends to $g(x, y)$ at almost every point (x, y). Therefore the relation

$$\sigma^*(x, y; f) = \limsup_{m, n\to\infty} |\sigma_{m,n}(x, y)| \le g(x, y)$$

holds good almost everywhere. Combining this result with (5.1) we obtain

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sigma^*(x, y; f)dxdy \le 2\pi A \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| dxdy + 2\pi B. \tag{5.2}$$

Let λ be a positive constant. Substituting λf for f in (5.2), we obtain

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sigma^*(x, y; f)dxdy \le 2\pi A \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| dxdy + 2\pi \frac{B}{\lambda}\cdot \tag{5.3}$$

Let ϵ be a positive number; we take λ so large that $2\pi B/\lambda < \epsilon/2$. Let

$$f(x, y) = f'(x, y) + f''(x, y)$$

be such that f' is bounded;

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f''(x, y)| \, dxdy < \epsilon, \tag{5.4}$$

$$2\pi A \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f''(x, y)| \log^+ |\lambda f''(x, y)| \, dxdy + 2\pi \frac{B}{\lambda} \cdot \tag{5.5}$$

Applying the inequality (5.3) to the function $f''(x, y)$, we obtain

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sigma^*(x, y; f'')dxdy < \epsilon$$

by observing (5.5). Combining this relation with (5.4), we see that the set $E(\epsilon)$ of points (x, y) such that either $|f''(x, y)| > \epsilon^{1/2}$ or $\sigma^*(x, y; f) > \epsilon^{1/2}$ is of plane measure less than $2\epsilon^{1/2}$. Now let $\sigma'_{\mu,\nu}$ and $\sigma''_{\mu,\nu}$ denote respectively the (μ, ν)th Fejér sums of the Fourier series of f' and f'', then

$$\left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |\sigma_{\mu,\nu} - f|^k\right) \leqq \left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |\sigma'_{\mu,\nu} - f|^k\right)^{1/k} + \left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |\sigma''_{\mu,\nu} - f''|^k\right)^{1/k}.$$

The first term on the right-hand side is $o(mn)^{1/k}$ almost everywhere, by Lemma 4. And

$$\left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |\sigma''_{\mu,\nu} - f''|^k\right)^{1/k} \leqq \left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |\sigma''_{\mu,\nu}|^k\right)^{1/k} + \left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |f''|^k\right)^{1/k}$$
$$\leqq [(m + 1)(n + 1)]^{1/k}(\sigma^*(x, y; f'') + |f''|).$$

Hence, outside the set $E(\epsilon)$,

$$\limsup_{m,n\to\infty} \left\{\frac{1}{(m + 1)(n + 1)} \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |\sigma_{\mu,\nu} - f|^k\right\}^{1/k} \leqq \sigma^*(x, y; f'') + |f''| \leqq 2\epsilon^{1/2}.$$

Since ϵ is arbitrary, the theorem follows.

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