REPRESENTATION OF FOURIER INTEGRALS AS SUMS. I

R. J. DUFFIN

Let $\phi(x)$ be an arbitrary function and let the functions $F(x)$ and $G(x)$ be defined by the series:

$$F\left(x\left(\frac{\pi}{2}\right)^{1/2}\right) = \phi(x) - \frac{1}{3}\phi\left(\frac{x}{3}\right) + \frac{1}{5}\phi\left(\frac{x}{5}\right) - \cdots,$$

$$G\left(x\left(\frac{\pi}{2}\right)^{1/2}\right) = \frac{1}{x}\phi\left(\frac{1}{x}\right) - \frac{1}{x}\phi\left(\frac{3}{x}\right) + \frac{1}{x}\phi\left(\frac{5}{x}\right) - \cdots.$$

Then $G(x)$ is the Fourier sine *transform of $F(x)$; that is,*

$$G(x) = \left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty \sin xtF(t)dt.$$

The purpose of this paper is to give restrictions on $\phi(x)$ so that this relation is valid in some sense.

It will be shown here that the restrictions on $\phi(x)$ are closely related to restrictions which insure the existence and inversion of $\int_0^\infty \sin xt\phi(t)dt$. Well known and important cases for the inversion of the Fourier transform are: 1. $\phi(t)\subset L_2$, 2. $\phi(t)\subset L_1$, and 3. $\phi(t)$ of bounded variation. The analogous cases will be considered.

It is convenient to employ the following notation: $\operatorname{sn} x = \sin (\pi/2)x$, $\operatorname{cs} x = \cos (\pi/2)x$, and $\alpha_n = \sin (\pi/2)n$; $n = 0, 1, 2, \cdots$. Thus we are trying to justify the relation

$$\sum_1^\infty \frac{\alpha_n}{x}\phi\left(\frac{n}{x}\right) = \int_0^\infty \operatorname{sn} xt\sum_1^\infty \frac{\alpha_n}{n}\phi\left(\frac{t}{n}\right)dt.$$

We shall call this in what follows the sine transform. (No confusion should result from the fact that sn x has a different meaning in the theory of elliptic functions.)

The proofs given here do not assume any previous knowledge of Fourier integrals although certain elementary properties of Fourier series are employed.

1. L_2 **theory.** We make the restriction on $\phi(x)$ not only that it belong to L_2 but also that at least one of the series converges suitably to a function in L_2.

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THEOREM 1a. *Let $\phi(x) \subset L_2$ $(0, \infty)$ and suppose as $N \to \infty$ that $\sum (\alpha_n/n)\phi(x/n)$, where n runs from 1 to N, converges in mean square to a function $f(x)$ in $L_2(0, \infty)$. Then $\sum (\alpha_n/x)\phi(n/x)$ converges in mean square to a function $g(x)$ in $L_2(0, \infty)$ and*

$$g(x) = \frac{d}{dx}\frac{2}{\pi}\int_0^\infty \frac{1 - \text{cs } xt}{t} f(t)dt$$

almost everywhere.

PROOF. We shall employ a simple lemma of some interest in itself.

LEMMA 1. *Let $\phi(x) \subset L_2(0, \infty)$ and suppose for some sequence of constants $\{b_n\}$ that as $N \to \infty$, $\sum (b_n/n)\phi(x/n)$, where n runs from 1 to N, converges in mean square to $f(x)$, $L_2(0, \infty)$. Then $\sum (b_n/x)\phi(n/x)$ converges in mean square to a function $g_1(x)$ in $L_2(0, \infty)$ and $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |g_1(x)|^2 dx$.*

If $tx = n\nu$ then

$$\begin{aligned}\int_0^\infty \left| \sum_{N_1}^{N_2} \frac{b_n}{x} \phi\left(\frac{n}{x}\right) \right|^2 dx &= \sum_{N_1}^{N_2}\sum_{N_1}^{N_2} \int_0^\infty \frac{b_n b_\nu}{x^2} \phi\left(\frac{n}{x}\right)\phi\left(\frac{\nu}{x}\right) dx \\ &= \sum_{N_1}^{N_2}\sum_{N_1}^{N_2} \int_0^\infty \frac{b_n b_\nu}{n\nu} \phi\left(\frac{t}{\nu}\right)\phi\left(\frac{t}{n}\right) dt \\ &= \int_0^\infty \left| \sum_{N_1}^{N_2} \frac{b_n}{n} \phi\left(\frac{t}{n}\right) \right|^2 dt.\end{aligned}$$

But as N_1 and N_2 approach ∞ the latter integral approaches zero.

LEMMA 2. *For $0 \leqq t \leqq 1$; $N = 1, 2, 3, \cdots$, the partial sums $\sum_1^N \alpha_n(1 - \cos nt)/nt$ are uniformly bounded.*

Let

$$S(x) = \sum_1^N \alpha_n \frac{1 - \cos nxt}{nt},$$

so

$$S'(x) = \sum_1^N \alpha_n \sin nxt.$$

But $\sin\theta - \sin 3\theta + \cdots \pm \sin(2m+1)\theta = (-1)^m \sin(2m+2)\theta/2\cos\theta$. So $S'(x) = \pm \sin N'xt/2 \cos xt$, $0 \leqq x \leqq 1$, where N' is an integer. By the mean value theorem $S(1) = S(0) + S'(x_1)$, $0 < x_1 < 1$. Since $S(0) = 0$ we have $|S(1)| \leqq 1/2 \cos 1$.

Let

$$Y = \frac{2}{\pi}\int_0^\infty \frac{1 - \operatorname{cs} xt}{t} f(t)dt = \lim_{N\to\infty} \frac{2}{\pi}\int_0^\infty \frac{1 - \operatorname{cs} xt}{t} \sum_1^N \frac{\alpha_n}{n} \phi\left(\frac{t}{n}\right) dt$$
$$= \lim_{N\to\infty} \frac{2}{\pi}\int_0^\infty \sum_1^N \alpha_n \frac{1 - \operatorname{cs} nxt}{n} \phi(t) \frac{dt}{t}\cdot$$

Split this integral into two parts, $\int_0^K + \int_K^\infty$; $K = 2/\pi x$.

The passage to the limit under the integral sign is justified in the first integral by Lemma 2. In the second integral the series is known to be uniformly bounded as it is the Fourier series of a function of bounded variation. Since $\int_K^\infty |\phi(t)| dt/t$ exists, passage to the limit under the integral sign is justified in the second integral. As is well known,

$$\frac{2}{\pi}\sum_1^\infty \alpha_n \frac{1 - \operatorname{cs} nx}{n} = \begin{cases} 0; & 4m - 1 < x < 4m + 1, \\ 1; & 4m + 1 < x < 4m + 3, \\ 1/2; & 2m + 1 = x. \end{cases}$$

Let

$$h(x) = \begin{cases} 0; & |x| > 1, \\ 1; & |x| < 1, \\ 1/2; & |x| = 1. \end{cases}$$

With $h(x)$ so defined it is obvious that the following key lemma is valid:

LEMMA 3. $(2/\pi)\sum\alpha_n(1 - \operatorname{cs} nx)/n = \sum\alpha_n h(n/x)$, *where n runs from* 1 *to* ∞ *in both sums.*

Thus, using this relation, we obtain

$$Y = \int_0^\infty \sum_1^\infty \alpha_n h\left(\frac{n}{tx}\right)\phi(t)\frac{dt}{t}\cdot$$

The partial sums of this series are bounded by 1 and are zero near the origin so

$$Y = \sum_1^\infty \int_0^\infty \alpha_n h\left(\frac{n}{tx}\right)\phi(t)\frac{dt}{t} = \sum_1^\infty \int_0^x \frac{\alpha_n}{t}\phi\left(\frac{n}{t}\right)dt = \int_0^x g(t)dt.$$

This last step is justified by Lemma 1. Thus $g(x) = dY/dx$ almost everywhere.

THEOREM 1b. *Under the hypotheses of Theorem* 1a *almost everywhere*

$$f(x) = \frac{d}{dx}\frac{2}{\pi}\int_0^\infty \frac{1 - \operatorname{cs} xt}{t} g(t)dt.$$

Let $\Phi(x)=\phi(1/x)/x$; then, by Lemma 1, Φ satisfies Theorem 1a. But

$$\sum_1^N \frac{\alpha_n}{n}\phi\left(\frac{x}{n}\right) \equiv \sum_1^N \frac{\alpha_n}{x}\Phi\left(\frac{n}{x}\right)$$

and

$$\sum_1^N \frac{\alpha_n}{x}\phi\left(\frac{n}{x}\right) \equiv \sum_1^N \frac{\alpha_n}{n}\Phi\left(\frac{x}{n}\right).$$

This observation completes the proof without recourse to Plancherel's theorem.

2. L_1 **theory.** Besides assuming that $\phi(x)\subset L_1$ we also assume that $\phi(x)$ is "small" near the origin.

THEOREM 2a. *Let* $(1+1/x)\phi(x)\subset L_1(0,\ \infty)$; *then*

$$f(x) = \sum_1^\infty \frac{\alpha_n}{n}\phi\left(\frac{x}{n}\right) \quad \textit{and} \quad g(x) = \sum_1^\infty \frac{\alpha_n}{x}\phi\left(\frac{n}{x}\right)$$

exist for almost all x and

$$g(x) = \frac{d}{dx}\frac{2}{\pi}\int_0^{\to\infty} \frac{1-\operatorname{cs} xt}{t} f(t)dt$$

almost everywhere.

Since $\phi(x)$ and $\phi(1/x)/x\subset L_1(0,\ \infty)$ the first part of the theorem is a consequence of this lemma:

LEMMA 4. *Let* $f(x)\subset L_1(1,\ \infty)$ *and let* $\{b_n\}$ *be a sequence of constants such that* $|b_n|<B$. *Then* $\sum b_n f(nx)$, *where n runs from* 1 *to* ∞, *exists for almost all* $x>1$ *and* $\int_1^\infty |\sum_1^\infty b_n f(nx)|dx/x \leqq B\int_1^\infty |f(x)|dx$.

Note that $\sum_1^N|f(nx)|$ and $\sum_1^N h(n/x)$ are monotone in N so by the Lesbegue integral theory of monotone sequences

$$\begin{aligned}\int_1^\infty \sum_1^\infty |f(nx)|\frac{dx}{x} &= \sum_1^\infty \int_1^\infty |f(nx)|\frac{dx}{x} \\ &= \sum_1^\infty \int_0^\infty |f(x)|\, h(n/x)\frac{d\alpha}{x} \\ &= \int_0^\infty |f(x)|\sum_1^\infty h(n/x)\frac{dx}{x} \leqq \int_1^\infty |f(x)|\,dx.\end{aligned}$$

Thus $\sum_1^\infty|f(nx)|$ converges for almost all x.

Since $|\sum_1^\infty b_n f(nx)| \leqq B\sum_1^\infty|f(nx)|$, the proof is complete.

The lemma implies $\int_0^\lambda \sum_1^\infty |\alpha_n\phi(t/n)/n|dt$ exists so

$$\frac{2}{\pi}\int_0^\lambda \frac{1-\text{cs } xt}{t}\sum_1^\infty \frac{\alpha_n}{n}\phi\left(\frac{t}{n}\right)dt$$
$$= \lim_{N\to\infty}\frac{2}{\pi}\int_0^\infty \sum_1^N \alpha_n \frac{1-\text{cs } nxt}{n} h\left(\frac{nt}{\lambda}\right)\phi(t)\frac{dt}{t}\cdot$$

For each value of t the series is some partial sum of the series $\sum_1^\infty \alpha_n(1-\text{cs } nxt)/n$ and since these partial sums are bounded the limit may be taken under the integral sign, giving

$$\frac{2}{\pi}\int_0^\infty \sum_1^\infty \alpha_n \frac{1-\text{cs } nxt}{n} h\left(\frac{nt}{\lambda}\right)\phi(t)\frac{dt}{t}\cdot$$

For the same reason the limit of this integral as $\lambda\to\infty$ is

$$\frac{2}{\pi}\int_0^\infty \sum_1^\infty \alpha_n \frac{1-\text{cs } nxt}{n}\phi(t)\frac{dt}{t} = \int_0^\infty \sum_1^\infty \alpha_n h\left(\frac{n}{xt}\right)\phi(t)\frac{dt}{t}$$
$$= \sum_1^\infty \int_0^\infty \alpha_n h\left(\frac{n}{xt}\right)\phi(t)\frac{dt}{t}$$
$$= \sum_1^\infty \int_0^x \frac{\alpha_n}{t}\phi\left(\frac{n}{t}\right)dt$$
$$= \int_0^x \sum_1^\infty \frac{\alpha_n}{t}\phi\left(\frac{n}{t}\right)dt.$$

The interchange of limits is justified by the dominated convergence of the series.

THEOREM 2b. *Under the hypotheses of Theorem* 2a *almost everywhere*

$$f(x) = \frac{d}{dx}\frac{2}{\pi}\int_0^{\to\infty}\frac{1-\text{cs } xt}{t}g(t)dt.$$

Let $\Phi(x)=\phi(1/x)/x$; then if $xt=1$,

$$\int_0^\infty |\Phi(x)|(1+1/x)dx = \int_0^\infty |\phi(t)|(1+1/t)dt.$$

Thus $\Phi(x)$ satisfies the conditions of Theorem 2a but

$$\sum_1^\infty \frac{\alpha_n}{n}\Phi\left(\frac{x}{n}\right) \equiv \sum_1^\infty \frac{\alpha_n}{x}\phi\left(\frac{x}{n}\right)$$

and

$$\sum_1^\infty \frac{\alpha_n}{x}\Phi\left(\frac{n}{x}\right) \equiv \sum_1^\infty \frac{\alpha_n}{n}\phi\left(\frac{n}{x}\right).$$

This observation completes the proof.

3. **Bounded variation theory.** Here our restrictions on $\phi(x)$ are weaker than the restrictions usually stated for the existence and inversion of the Fourier transform in the corresponding case. The Stieltjes integral is found to be the natural tool for the proof in this case. The more transcendental Lesbegue integral is not needed at all.

THEOREM 3a. *Let $x\phi(x)/1+x$ be of bounded variation in $(0, \infty)$ and tend to zero at zero and infinity. Then*

$$f(x) = \sum_1^\infty \frac{\alpha_n}{2n}\left\{\phi\left(\frac{x}{n}+\right) + \phi\left(\frac{x}{n}-\right)\right\}$$

and

$$g(x) = \sum_1^\infty \frac{\alpha_n}{2x}\left\{\phi\left(\frac{n}{x}+\right) + \phi\left(\frac{n}{x}-\right)\right\}$$

are convergent for positive x and $g(x) = \int_0^{\to\infty}$ sn $xtf(t)dt$.

Without loss of generality we shall suppose that $\phi(x)$ is normalized; that is, $\phi(x+)+\phi(x-)=2\phi(x)$. For any positive a, $\phi(x)$ is of bounded variation in (a, ∞) and $x\phi(x)$ is of bounded variation in $(0, a)$. Therefore $\phi(x)=\phi_1(x)-\phi_2(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are positive and monotone and tend to 0 at ∞. Clearly then

$$\left|\sum_1^\infty \frac{\alpha_n}{x}\,\phi\left(\frac{n}{x}\right)\right| \leqq \frac{1}{x}\phi_1\left(\frac{1}{x}\right) + \frac{1}{x}\phi_2\left(\frac{1}{x}\right).$$

Also $x\phi(x)=\psi_1(x)+\psi_2(x)$ where $\psi_1(x)$ and $\psi_2(x)$ are positive and monotone and tend to 0 at 0. Then

$$\left|\sum_1^\infty \frac{\alpha_n}{n}\,\phi\left(\frac{x}{n}\right)\right| \leqq \frac{1}{x}\psi_1(x) + \frac{1}{x}\psi_2(x).$$

This proves the first part of the theorem. To prove the second part, first suppose $\phi(x)$ constant for $0\leqq x<\beta\leqq 1/2$, then

$$\begin{aligned}\int_0^\lambda \text{sn}\, t\sum_1^N \frac{\alpha_n}{n}\,\phi\left(\frac{t}{n}\right)dt &= \int_0^\infty \sum_1^N \alpha_n \,\text{sn}\, tnh\left(\frac{nt}{\lambda}\right)\phi(t)dt \\ &= -\int_0^\infty \int_0^t \sum_1^N \alpha_n \,\text{sn}\, pnh\left(\frac{np}{\lambda}\right) dp d\phi(t) \\ &= -\int_{\beta-}^\infty \int_0^t \sum_1^N \alpha_n \,\text{sn}\, pnh\left(\frac{np}{\lambda}\right) dp d\phi(t).\end{aligned}$$

Suppose $N\geqq\lambda/\beta$ and $t\geqq\beta$; then $N>[\lambda/t]$ so

$$\int_0^t \sum_1^N \alpha_n \operatorname{sn} pnh\left(\frac{np}{\lambda}\right) dp = \frac{2}{\pi} \sum_1^{[\lambda/t]} \alpha_n \frac{1 - \operatorname{cs} nt}{n} + (1 - \operatorname{cs} \lambda) \frac{2}{\pi} \sum_{[\lambda/t]+1}^{N} \frac{\alpha_n}{n}.$$

This series converges uniformly to a continuous uniformly bounded function $Q_\lambda(t)$ as $N \to \infty$. Therefore

$$\int_0^\lambda \operatorname{sn} t \sum_1^\infty \frac{\alpha_n}{n} \phi\left(\frac{t}{n}\right) dt = -\int_0^\infty Q_\lambda(t) d\phi(t).$$

As $\lambda \to \infty$, $Q_\lambda(t)$ is uniformly bounded and approaches $\sum_1^\infty \alpha_n 2 \cdot (1 - \operatorname{cs} nt)/\pi n$ uniformly, excepting in neighborhoods of the odd integer points. Let $\phi(t) = \phi_c(t) + \phi_s(t)$ where $\phi_c(t)$ is continuous and $\phi_s(t)$ is a step function, constant except at the odd integer points.

$$-\int_0^\infty Q_\lambda(t) d\phi = -\int_0^\infty Q_\lambda(t) d\phi_c - \sum_1^\infty Q_\lambda(2n-1)\delta\phi_s(2n-1).$$

The limit of this expression as $\lambda \to \infty$ is

$$\sum_1^\infty \left\{\phi_c(4n-3) - \phi_c(4n-1)\right\} - \frac{1}{2}\sum_1^\infty \delta\phi_s(2n-1).$$

But

$$\sum_1^\infty \delta\phi_s(2n-1)$$

$$= \sum_1^\infty \left\{\phi_s(4n-1+) + \phi_s(4n-1-) - \phi_s(4n-3+) - \phi_s(4n-3-)\right\}.$$

Substituting this latter expression gives finally $\sum_1^\infty \alpha_n \phi(n)$.

LEMMA 5. *Suppose $\psi(x)$ is a nondecreasing function for $0 \leqq x < \beta \leqq 1/2$, $\psi(x) = 0$ for $x > \beta$, and $\psi(0+) = 0$. Then*

$$\left| \int_0^\lambda \frac{\operatorname{sn} t}{t} \sum_1^\infty \alpha_n \psi\left(\frac{t}{n}\right) dt \right| \leqq A\psi(\beta -)$$

where A is independent of λ and β.

Let N be such that for a given λ

$$\left| \int_0^\lambda \frac{\operatorname{sn} t}{t} \sum_{N+1}^\infty \alpha_n \psi\left(\frac{t}{n}\right) dt \right| \leqq \psi(\beta -).$$

Now

$$\int_0^\lambda \frac{\operatorname{sn} t}{t} \sum_1^N \alpha_n \psi\left(\frac{t}{n}\right) dt = \int_0^\infty \psi(t) \sum_1^N \alpha_n \operatorname{sn} tnh\left(\frac{tn}{\lambda}\right) \frac{dt}{t}$$

$$= \psi(\beta -) \int_\theta^\beta \sum_1^N \alpha_n \operatorname{sn} tnh\left(\frac{tn}{\lambda}\right) \frac{dt}{t} \cdot$$

The second law of the mean defines θ. If $N \leqq \lambda/\beta$ then

$$\int_0^\beta \sum_1^N \alpha_n \operatorname{sn} tnh\left(\frac{tn}{\lambda}\right) \frac{dt}{t} = \int_0^\beta \sum_1^N \alpha_n \operatorname{sn} tn \frac{dt}{t}$$

$$= \pm \int_0^\beta \frac{\operatorname{sn} N't}{2 \operatorname{cs} t} \frac{dt}{t}$$

$$= \pm \frac{1}{2 \operatorname{cs} \beta} \int_{\theta_2}^\beta \operatorname{sn} N't \frac{dt}{t} \cdot$$

This last expression is bounded independently of N, λ, and β. If $N > \lambda/\beta$ then a similar evaluation gives

$$\pm \int_0^\beta \frac{\operatorname{sn} M't}{2 \operatorname{cs} t} \frac{dt}{t} + \int_0^\lambda \frac{\sin t}{t} dt \sum_{M+1}^N \alpha_n; \qquad M = [\lambda/\beta].$$

This expression is also bounded independently of N, λ, and β. The same considerations apply to the integral $\int_0^\theta$ and the lemma is proved.

We are now able to handle the general case. Define $\phi_i(x) + \phi_e(x) = \phi(x)$ where $\phi_i(x) = 0$ for $x < \beta$ and $\phi_e(x) = 0$ for $x > \beta$. Then $x\phi_e(x) = \psi_1(x) - \psi_2(x)$ where $\psi_1(x)$ and $\psi_2(x)$ satisfy the conditions of Lemma 5.

Let β satisfy $A\psi_1(\beta) \leqq \epsilon$, $A\psi_2(\beta) \leqq \epsilon$, and $0 < \beta \leqq 1/2$. There is a λ_0 such that for $\lambda \geqq \lambda_0$

$$\left| \int_0^\lambda \operatorname{sn} t \sum_1^\infty \frac{\alpha_n}{n} \phi_i\left(\frac{t}{n}\right) dt - \sum_1^\infty \alpha_n \phi(n) \right| < \epsilon.$$

Therefore by Lemma 5

$$\left| \int_0^\lambda \operatorname{sn} t \sum_1^\infty \frac{\alpha_n}{n} \phi\left(\frac{t}{n}\right) dt - \sum_1^\infty \alpha_n \phi(n) \right| < 3\epsilon.$$

The observation that $\phi(ct)$ for $c > 0$ satisfies the conditions of the theorem completes the proof.

THEOREM 3b. *If $\phi(x)$ satisfies the conditions of Theorem 3a then $f(x) = \int_0^{\to\infty} \operatorname{sn} xtg(t)dt$.*

Let $\Phi(x)=\phi(1/x)/x$. Then $x\Phi(x)/1+x=\phi(1/x)/1+x$, so clearly $\Phi(x)$ satisfies the conditions of Theorem 3a which implies Theorem 3b for $\phi(x)$.

4. **Remarks.** It will be noted that the proofs for the three cases are independent; however, the methods of proof are similar. The three cases are given in the apparent order of increasing difficulty of proof.

If $x\phi(x)\equiv\phi(1/x)$ then $f(x)\equiv g(x)$ and we arrive at a representation of self-reciprocal functions as sums.

While much has been written on self-reciprocal functions, not so much attention has been given to function spaces or classes which are self-reciprocal as a whole. The notable exception to this statement is, of course, the space L_2. Note then, that the functions $f(x)$ and $g(x)$ defined by Theorem 2a belong to the same linear function space. Again the functions $f(x)$ and $g(x)$ of Theorem 3a belong to another linear function space.

The theorems here show that under quite general conditions two operations of different form give rise to the same functional transformation. It is natural then to attempt to extend the range of the transformation by using either operation alone where both are not applicable. For example, if $\phi(x)$ is different from zero at only one point the series for $f(x)$ and $g(x)$ are, of course, convergent and $g(x)$ may be defined as the "Fourier transform" of $f(x)$. However since $f(x)$ and $g(x)$ are zero except in a set of measure zero the ordinary integral definition of the Fourier transform is silent since a set of measure zero is disregarded in integration.

The following examples of the theory are of some interest. If $\phi(x)=1/(x+1)$ then

$$f(x) = g(x) = 1/(x+1) - 1/(x+3) + 1/(x+5) - \cdots;$$

clearly $g(x)=\int_0^1(t^x/(1+t^2))dt$.

Again if $\phi(x)=x^{-s}$, $0<s<1$, then $f(x)=x^{-s}L(1-s)$ and $g(x)=x^{s-1}L(s)$. Here $L(s)=1-3^{-s}+5^{-s}-\cdots$. Evaluation of the integral $\int_0^\infty \operatorname{sn} xtf(t)dt$ gives the well known identity, $L(s)=(\pi/2)^s\cdot\cos(s\pi/2)\Gamma(1-s)L(1-s)$.

The second part of this paper treats other conditions on $\phi(x)$ and also the direct representation of the sine transform as a double sum

$$g(x) = \sum_1^\infty\sum_1^\infty \frac{\mu_n\alpha_n\alpha_m}{nx} f\left(\frac{n}{mx}\right).$$

Here μ_n is the well known Möbius symbol.

UNIVERSITY OF ILLINOIS