HADAMARD'S THREE CIRCLES THEOREM

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Hadamard's theorem is concerned with the relation between the maximum absolute values of an analytic function on three concentric circles.[1] If we put

$$M(r) = \max_{|z|=r} |f(z)|,$$

then the theorem states that $\log M(r)$ is a convex function of $\log r$ for $r'<r<r''$, if $f(z)$ is regular for $r'<|z|<r''$. This is an immediate consequence of the fact that if $|f(z)| \leqq A|z|^\lambda$ on two circles about the origin, then it is also true between the circles; and this in turn is seen by applying the principle of maximum to $f(z)/z^\lambda$. The bound is attainable within the ring only for $f(z) = \alpha z^\lambda$ with $|\alpha| = A$. Notice that this function is single-valued only if λ is an integer, so that Hadamard's bound is not in general sharp for single-valued functions. (It is the sharp bound for the class of many-valued functions, any branch of which is regular in the ring, and for which $|f(z)|$ is single-valued.)

We shall consider only single-valued functions. The problem of finding the sharp bound in Hadamard's theorem is formulated as Problem A below. (It is no essential restriction to suppose that the radius of the outer circle is 1, and that the given bound on this circle is 1.) Problems B and C raise the same question for more special classes of functions.

PROBLEM A. *Suppose* $0<q<Q<1$ *and* $p>0$. *Consider the class of functions satisfying the following conditions*: $f(z)$ *is regular for* $q \leqq |z| \leqq 1$,

$$|f(z)| \leqq 1 \quad \textit{for} \quad |z| = 1, \qquad |f(z)| \leqq p \quad \textit{for} \quad |z| = q.$$

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[1] The theorem was stated (without proof) in Hadamard's note, *Sur les fonctions entières*, Bull. Soc. Math. France vol. 24 (1896) pp. 186–187. His proof was apparently first published in 1912; it may be found in footnote 2, p. 94, of *Selecta*: *Jubilé Scientifique de M. Jacques Hadamard*, Paris, 1935. In the meantime, proofs (of a less simple nature) had been given by O. Blumenthal and by G. Faber. See Blumenthal, *Über ganze transzendente Funktionen*, Jber. Deutschen Math. Verein. vol. 16 (1907) pp. 97–109, and *Sur le mode de croissance des fonctions entières*, Bull. Soc. Math. France vol. 35 (1907) pp. 213–232; Faber, *Über das Anwachsen analytischer Funktionen*, Math. Ann. vol. 63 (1907) pp. 549–551.

Let P be the largest value of $|f(Q)|$ for any function of the given class. How much is P, and for what functions is it attained?

PROBLEM B. *The same as Problem* A, *with the additional hypothesis that the coefficients of the Laurent series for $f(z)$ are positive or zero.*

PROBLEM C. *The same as Problem* A, *with the additional hypotheses that $p<1$ and that $f(z)$ is regular also for $|z|<q$ (hence for $|z|\leq 1$).*

REMARKS. From any function $f(z)$ with $|f(Q)|=P$ we can obtain a function $H(z)$ with $H(Q)=P$, by putting $H(z)=Pf(z)/f(Q)$. The name *extremal function* will be applied only to an admissible function with $H(Q)=P$.

We may determine a real λ such that $q^\lambda=p$. If λ is an integer, then $H(z)=z^\lambda$ is the extremal function for all three problems; but if λ is not an integer, then z^λ is not an admissible function. We may restrict ourselves to the latter case, and shall use n to denote the integer such that $n-1<\lambda<n$.

If we indicate the dependence of the extremal function on p by using the notation $H(z, p)$, then it is clear that for the first two problems we have

$$H(z, pq^k) = z^k H(z, p)$$

for any integer k, since a power of z times an admissible function is admissible; but for Problem C no such relation is to be expected. Consequently, there is no loss of generality in supposing $q<p<1$ (that is, $n=1$) when studying Problems A and B.

Summary of results.[2] We state here some of the principal results that are known concerning the three problems. For each of the problems, the extremal function $H(z)$ exists and is unique, and is real for real z. It is univalent if $q<p<1$. In Problems A and B, $H(z)$ is independent of Q; and the same is true in Problem C at least if $q<p<1$. We tabulate some additional results in the three cases for comparison.

PROBLEM A.

$zH'(z)/H(z)$ is an elliptic function of $\log z$.

[2] Problem A was first solved by O. Teichmüller, *Eine Verschärfung des Dreikreisesatzes*, Deutsche Mathematik vol. 4 (1939) pp. 16–22. But we shall follow here the solution given by the author in *Analytic functions in circular rings*, Duke Math. J. vol. 10 (1943) pp. 341–354. Problem B was solved by F. Carlson, *Sur le module maximum d'une fonction analytique uniforme*, Arkiv för Mathematik, Astronomi, och Fysik vol. 26A (1938). Problem C is studied by M. H. Heins in a paper, *On a problem of Walsh concerning the Hadamard three circles theorem*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 349–372, which I had the privilege of reading before publication.

$|H(z)|=1$ for $|z|=1$.
$|H(z)|=p$ for $|z|=q$.
If $q<p<1$, $w=H(z)$ maps $q<|z|<1$ on $|w|<1$ omitting an arc of $|w|=p$.

PROBLEM B.
$H(z)$ is an average of z^{n-1} and z^n.
$|H(z)|<1$ for $|z|=1$ except at $z=1$.
$|H(z)|<p$ for $|z|=q$ except at $z=q$.
If $q<p<1$, $w=H(z)$ is a contraction with 1 as fixed point.

PROBLEM C.
$H(z)$ is a rational function of the nth degree.
$|H(z)|=1$ for $|z|=1$.
$|H(z)|<p$ for $|z|=q$ except at n points.
If $q<p<1$, $w=H(z)$ maps the unit circle onto itself.

Study of Problem A. We suppose $q<p<1$. It may be seen from general mapping theorems that there exists a function $H(z)$ which maps the ring $q<|z|<1$ on $|w|<1$ omitting an arc of $|w|=p$. We may suppose that p is the midpoint of the omitted arc. The function $H(z)$ is regular on the boundaries of the ring, and we have

$$|H(z)| = 1 \quad \text{for} \quad |z| = 1, \qquad |H(z)| = p \quad \text{for} \quad |z| = q.$$

If $f(z)$ is any admissible function, then $|f(z)/H(z)| \leqq 1$ on both boundaries of the ring. We could apply the principle of maximum to conclude that $|f(Q)/H(Q)| \leqq 1$, were it not for the fact that $H(z)$ has a zero in the ring, so that $f(z)/H(z)$ has a pole. The fundamental lemma of the author's paper provides an extension of this principle which enables the conclusion to be drawn nevertheless. The lemma states that *if a function is regular in a circular ring except for one simple pole, and does not exceed* 1 *in absolute value on the boundaries, then it is less than* 1 *on the radius opposite the pole.* Applying this lemma, we verify that $H(z)$ is the desired extremal function.

By applying Schwarz's reflection principle, we can continue $H(z)$ to the whole plane excluding 0 and ∞. The reflections on the outer and inner boundaries give the relations

$$H(1/z) = 1/H(z), \qquad H(q^2/z) = p^2/H(z),$$

if we use the fact that $H(z)$ is real for real z. From these it follows that

$$H(q^2 z) = p^2 H(z),$$

and hence

$$q^2zH'(q^2z)/H(q^2z) = zH'(z)/H(z).$$

Thus $zH'(z)/H(z)$ is an elliptic function of $\log z$ with the periods $2 \log q$ and $2\pi i$.

It is not difficult to obtain the explicit formula

$$H(z) = z\theta(qz/p)/\theta(pz/q)$$

where

$$\theta(z) = \sum_{k=-\infty}^{\infty} q^{k^2}z^k.$$

This enables easy calculation of the extremal function.

Study of Problem B. Here

$$f(z) = \sum_{k=-\infty}^{\infty} c_kz^k$$

with $c_k \geqq 0$, hence $M(r) = f(r)$. Evidently

$$M''(r) \geqq 0,$$

the equality holding only if $M(r) = c_0 + c_1r$. If $q < p < 1$, we can determine positive c_0 and c_1 so that $M(1) = 1$, $M(q) = p$; that is, so that

$$c_0 + c_1 = 1, \qquad c_0 + c_1q = p.$$

With this determination of c_0 and c_1, the extremal function is

$$H(z) = c_0 + c_1z.$$

This result of Carlson, which concerns a special class of functions, has an interesting application to the more general class previously considered. In fact, if we no longer suppose that $f(z)$ has positive coefficients, we have nevertheless that the average of $|f(z)|^2$ on $|z| = r$ is

$$\sum_{k=-\infty}^{\infty} |c_k|^2 r^{2k},$$

which is a power series in r^2 with positive coefficients, so that Carlson's result may be applied. If we suppose given that the average of $|f(z)|^2$ on $|z| = 1$ does not exceed 1, and that the average on $|z| = q$ does not exceed p^2 $(q < p < 1)$, then the function $H(z)$ having the largest quadratic mean on $|z| = Q$ is of the form

$$H(z) = c_0 + c_1z,$$

where c_0 and c_1 are subject to the conditions

$$|c_0|^2 + |c_1|^2 = 1, \qquad |c_0|^2 + |c_1|^2 q^2 = p^2.$$

Some lemmas. We consider now some results that are used in the study of Problem C. The results concern functions constant in absolute value on a circle, and interpolation by bounded functions.

In the first place, an equality such as $|F(z)| = 1$ either holds identically on $|z| = 1$ or at but a finite number of points, provided $F(z)$ is regular on $|z| = 1$. For on the circle, $|F(z)| = 1$ is equivalent to

$$F(z)\overline{F}(1/z) = 1.$$

Since the left side is regular on the circle, the result follows.

Suppose now that $F(z)$ is regular for $|z| \leqq 1$ and that $|F(z)| = 1$ for $|z| = 1$. Let $a_1, a_2, \cdots, a_n$ be the zeros of $F(z)$ in $|z| < 1$. Then we find that

$$F(z) = \alpha \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z} \qquad (|\alpha| = 1)$$

by applying the principle of maximum and minimum to $F(z)$ divided by the product on the right. The zeros and poles of $F(z)$ are inverse with respect to the given circle $|z| = 1$. A similar result holds for any other circle. If $|F(z)|$ were constant on two circles about the origin, $F(z)$ being regular within the larger circle, then the zeros and poles would have to be inverse with respect to both circles, that is, the zeros at 0 and the poles at ∞, and hence $F(z) = \alpha z^n$.

Concerning interpolation by bounded functions, we need the following theorem. *Let $z_1, z_2, \cdots, z_n, \zeta$ be $n+1$ distinct points in $|z| < 1$, and let $w_1, w_2, \cdots, w_n, \omega$ be any $n+1$ points in $|w| \leqq 1$. Consider the class of functions $F(z)$ regular for $|z| < 1$ and with $|F(z)| \leqq 1$ there. The number of such functions satisfying the interpolating conditions*

$$F(z_k) = w_k \qquad (k = 1, 2, \cdots, n)$$

may be 0, 1, *or* ∞. *If there is just one such function, then it is rational of less than the nth degree, and satisfies $|F(z)| = 1$ for $|z| = 1$. If there are infinitely many such functions, then the possible values of $F(\zeta)$ fill a closed circle; the additional condition*

$$F(\zeta) = \omega$$

will determine the function uniquely if and only if ω is on the boundary of that circle.

The proof is by induction. Consider first the case $n = 0$. The number of functions is infinite. The possible values of $F(\zeta)$ fill the circle $|w| \leqq 1$. The condition $F(\zeta)=\omega$ determines $F(z)$ uniquely, if and only if $|\omega|=1$.

Suppose now that $n>0$. In case $|w_n|=1$, if there is any solution it is $F(z)=w_n$, which is a rational function of the zeroth degree and satisfies $|F(z)|=1$ for $|z|=1$. In case $|w_n|<1$, we establish a one-to-one correspondence between the given functions and those satisfying certain interpolating conditions at the points $z_1, \cdots, z_{n-1}$ by means of the equation

$$G(z) = \frac{F(z) - w_n}{1 - \bar{w}_n F(z)} : \frac{z - z_n}{1 - \bar{z}_n z} \cdot$$

The various desired conclusions about $F(z)$ follow easily from the corresponding conclusions about $G(z)$.

Study of Problem C. Here the case $q<p<1$ has a very simple solution (but the general case cannot be reduced to it). For if the condition $|f(z)| \leqq p$ for $|z|=q$ is replaced by the weaker condition $|f(q)| \leqq p$, then it follows (using Schwarz's lemma) that the maximum possible value of $|f(Q)|$ is attained by the linear function $H(z)$ which maps the unit circle onto itself, with ± 1 as fixed points, and $H(q)=p$. But this function clearly satisfies $|H(z)| \leqq p$ for $|z|=q$ (under the hypothesis $q<p<1$), and hence is the required extremal function.

For the general case, Heins reaches his results by a rather indirect method. It will be convenient to modify the statement of Problem C by supposing that $f(z)$ is regular for $|z|<1$ rather than for $|z| \leqq 1$. The condition $|f(z)| \leqq 1$ for $|z|=1$ may be replaced by $|f(z)|<1$ for $|z|<1$. With this modification, the existence of an extremal function $H(z)$ is clear from the theory of normal families. It will be shown later that this function is unique, regular for $|z|=1$, and that $|H(z)|=1$ for $|z|=1$.

As a first step, we show that if $H(z)$ is regular for $|z|=1$, then $|H(z)|=1$ there. Otherwise, $|H(z)|=1$ at only a finite number of points on $|z|=1$, and hence we can find a small arc AB of $|z|=1$, near 1, where $|H(z)|<1$. Choose $K>1$ so that $K|H(z)|<1$ on this arc. Now it is easy to construct a function $g(z)$, regular for $|z| \leqq 1$ except at A and B, with constant absolute values on every circular arc joining A and B, these values varying from K on the given arc AB to 1 on the complementary arc of the unit circle. Since $|H(z)g(z)| \leqq 1$ for $|z|=1$, except at A and B, and $|H(z)g(z)|<1$ near these points, it follows that $|H(z)g(z)|<1$ for $|z|<1$. The function $H(z)g(z)$ may fail to be an admissible $f(z)$ by being too large on $|z|=q$. However,

$g(z)$ is larger at Q than on $|z|=q$, so that if we divide $H(z)g(z)$ by the maximum of $g(z)$ on $|z|=q$, we have a function $f(z)$ which is admissible and for which $|f(Q)|>|H(Q)|$, which is impossible.

Next we see that $|H(z)|=p$ has only a finite number of roots on $|z|=q$. For otherwise it would be an identity, and $H(z)$ would be rational. But then $H(z)$ would also be regular for $|z|=1$, and hence $|H(z)|=1$ there. From the fact that $|H(z)|$ is constant on two circles, we should conclude that $H(z)=z^n$, and hence $p=q^n$, which is the case we have excluded.

We shall now show that in the hypothesis that $|f(z)|\leqq p$ for $|z|=q$, only a finite number of points on $|z|=q$ have any weight. Let $H(z)$ be an extremal function, and let $z_1, z_2, \cdots, z_l$ be the points on $|z|=q$ where $|H(z)|=p$. Then if $|F(z)|<1$ for $|z|<1$, and $|F(z_k)|\leqq p$ $(k=1, 2, \cdots, l)$, we can conclude that $|F(Q)|\leqq P$ (that is, we obtain the same bound for $|F(Q)|$ as if we had supposed that $|F(z)|\leqq p$ for $|z|=q$). For if there were such a function with $|F(Q)|>P$, then we could also find a function with $|F(z_k)|<p$ and $F(Q)>P$. But then we see that

$$f(z) = (1-\epsilon)H(z) + \epsilon F(z)$$

is admissible (if ϵ is sufficiently small), and that $f(Q)>P$.

The next step is to see that $H(z)$ can be determined by interpolation. We shall show that if $|F(z)|<1$ for $|z|<1$, if $F(z_k)=H(z_k)$ $(k=1, 2, \cdots, l)$, and if $F(Q)=H(Q)$, then $F(z)=H(z)$ identically. Consider first the interpolating problem defined by $F(z_k)=H(z_k)$. The possible values of $F(Q)$ are restricted to a circle including P. If P were not on the boundary, then $F(Q)>P$ would be possible. Thus P is on the boundary, and the additional condition $F(Q)=P$ serves to determine $F(z)$ uniquely.

Thus $H(z)$ is a rational function of at most the lth degree, with $|H(z)|=1$ for $|z|=1$. From this the uniqueness of $H(z)$ follows at once. For if both $H_1(z)$ and $H_2(z)$ were extremal, then so also would be their average. This average must also satisfy the condition $|H(z)|=1$ for $|z|=1$, which is possible only if $H_1(z)=H_2(z)$ on $|z|=1$ and hence identically. Furthermore, if $H(z)$ is extremal, so also is $\overline{H}(z)$; hence $\overline{H}(z)=H(z)$, or $H(z)$ is real for real z.

Now consider the degree of $H(z)$. In the first place, $|H(z)|=p$ has l different roots on $|z|=q$, and these are all of even order, since $|H(z)|\leqq p$ on $|z|=q$. In other words, the equation

$$H(z)H(q^2/z) = p^2$$

has at least $2l$ roots, so that $H(z)$ cannot be of less than the lth degree. Thus $H(z)$ is exactly of the lth degree, and the roots of the displayed equation all lie on $|z| = q$, and are double roots.

As in the introduction, let n be the integer such that $n-1<\lambda<n$. We shall give a sketch of the proof that $l=n$.

In the first place, it is not difficult to show that P is a continuous function of q, p, Q; and using this fact, it may be shown that (for $|z| <1$) $H(z)$ depends continuously on these parameters.

Next we notice that l is a function of n only. For as long as we exclude the case in which λ is an integer, l is the number of double roots of $H(z)H(q^2/z)-p^2$ on $|z| =q$. Since this function is regular and depends continuously on the parameters for $q^2<|z| <1$, and has roots only on $|z| =q$, it cannot gain or lose a root.

Recalling that the degree of $H(z)$ is equal to the number of zeros in $|z| <1$, it is easy to see that the degree is a lower semi-continuous function of the parameters. Since the degree is n when $\lambda=n$, it cannot be less than n when λ is slightly less than n. Combined with the preceding result, this shows that $l\geqq n$.

Finally, by an ingenious method which we cannot consider here, Heins finds (for any given n) some cases in which it can be shown that the degree of $H(z)$ does not exceed n. This completes the proof that $H(z)$ is of exactly the nth degree when $n-1<\lambda<n$, and that $|H(z)| =p$ has exactly n different roots on $|z| =q$.

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