# CONTRACTIONS IN NON-EUCLIDEAN SPACES

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The existence of an extension of the range of definition of a function $f(x)$ defined on a set $S$ of a metric space $M$ to a metric space $M'$ so as to preserve a contraction of the type

$$\|f(x_1), f(x_2)\|' \leqq \|x_1, x_2\| \tag{1}$$

depends upon $M$ and $M'$. The author has previously shown [3, 4][1] that for $M = M'$ the extension exists when $M$ is: (1) the $n$-dimensional Euclidean space; (2) the surface of the $n$-dimensional Euclidean sphere; (3) the general Hilbert space. In this brief article *the extension is shown to exist when each $M$ and $M'$ is the $n$-dimensional hyperbolic space.* The method used to prove this result is applied to a metric space which includes both the hemispherical and hyperbolic cases. Hence a unification of results is also obtained.

As shown in the previous papers [3, 4] a necessary and sufficient condition for a contraction to be extensible in $M$ and $M'$ is the property E, which is restated as follows.

PROPERTY E. *Consider in each of the metric spaces $M$ and $M'$ a set of spheres, such that to each sphere $S_i \in M$, having center $x_i$ and radius $r_i$, there corresponds a sphere $S_i' \in M'$, having center $x_i'$ and radius $r_i'$. Furthermore suppose that*

$$r_i = r_i', \qquad \|x_i', x_j'\|' \leqq \|x_i, x_j\| \tag{2}$$

*for all corresponding spheres $S_i$ and $S_i'$, and for all corresponding pairs $(S_i, S_j)$ and $(S_i', S_j')$.*

*The spaces $M$ and $M'$ are said to have the extensibility property* E *if conditions* (2) *and*

$$\prod_i S_i \neq 0 \tag{3}$$

*imply that*

$$\prod_i S_i' \neq 0. \tag{4}$$

If the above statement holds for $M = M'$, the space $M$ is said to have property E.

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[1] Numbers in brackets refer to references at the end of the paper.

For convenience of discussion let $M$ be an $n$-dimensional metric space which can be imbedded in an $(n+1)$-dimensional Euclidean space $\mathcal{R}$. Let $R_i$ be the Euclidean vector emanating from the origin of $\mathcal{R}$ to the point $x_i \in M$. It is assumed that there exists a symmetric real-valued bilinear product $R_i \cdot R_j$ such that $R_i \cdot R_i = k^2 = \text{const.}$ defines the metric space $M$. Suppose that for any two points $x_i$ and $x_j$ in $M$, $R_i$ and $R_j$ determine a Euclidean plane which intersects $M$ in a *unique* continuous curve joining $x_i$ and $x_j$. This curve is defined to be a geodesic. Furthermore suppose the distance $\|x_i, x_j\|$ in $M$ is defined to be

$$\|x_i, x_j\| = F(R_i \cdot R_j) \geqq 0, \tag{5}$$

where $F(u)$ is either a single-valued *increasing* function of $u$ or a single-valued *decreasing* function of $u$.

THEOREM 1. *If the n-dimensional metric space M has the above properties, it possesses the property* E.

PROOF. To prove this we consider the case $F(u)$ is an *increasing* function of $u$. When $F(u)$ is decreasing the proof is obtained by a uniform change in the direction of the inequality signs. On account of a theorem of Helly[2] type [**2**, **1**], to prove Theorem 1 it is sufficient to establish property E for $i = 1, \cdots, n+1$.

Let $\Delta(x_1, \cdots, x_{n+1})$ be the simplex (degenerate or nondegenerate) in $M$ determined by the points $x_i$ $(i = 1, \cdots, n+1)$. Condition (4) implies that $\Delta(x_{i_1}, x_{i_2}) \cdot S_{i_1} \cdot S_{i_2} \neq 0$. If we have $\Delta(x_{i_1}, \cdots, x_{i_{r+1}}) \cdot \prod_{j=i_1}^{i_r} S_j \neq 0$ $(i = 1, \cdots, n+1;\ 2 \leqq r \leqq n)$, then since by (4) $\prod_{j=i_1}^{i_{r+1}} S_j \neq 0$, the theorem of Helly[3] type implies in the $r$-dimensional subspace that $\Delta(x_{i_1}, \cdots, x_{i_{r+1}}) \cdot \prod_{j=i_1}^{i_{r+1}} S_j \neq 0$. Hence

$$\Delta(x_1, \cdots, x_{n+1}) \cdot \prod_{i=1}^{n+1} S_i \neq 0 \tag{6}$$

is established by induction. *Suppose* that $\Delta(x_1', \cdots, x_{n+1}')$ is *not covered* by the spheres $S_i'$. Then choose $x$ and $x'$ so that

$$x \in \Delta(x_1, \cdots, x_{n+1}) \cdot \prod_{i=1}^{n+1} S_i, \quad x' \in \Delta(x_1', \cdots, x_{n+1}') - \sum_{i=1}^{n+1} S_i', \tag{7}$$

and let $R$ and $R'$ be the Euclidean vectors emanating from the origin

[2] The theorem states: *If each $n+1$ sets of a family of closed bounded, convex sets of the n-dimensional Euclidean space intersect, then there is a point common to all the sets.* For a more general topological theorem of the same type, see Alexandroff and Hopf [**1**, p. 297].

[3] Loc. cit.

of $\mathfrak{R}$ to the points $x$ and $x'$ respectively. Conditions (2) and (5) yield the results

$$R_i \cdot R_j \geqq R_i' \cdot R_j' \qquad (i, j = 1, \cdots, n+1). \tag{8}$$

Also conditions (7) and the first of conditions (2) imply that $\|x_i', x'\| > \|x_i, x\|$. Hence by (5) we have

$$R' \cdot R_i' > R \cdot R_i. \tag{9}$$

Since the line of shortest length joining $x_i$ and $x_j$ lies in the plane determined by $R_i$ and $R_j$, the simplex $\Delta(x_1, \cdots, x_{n+1})$ is contained in the smaller solid angle $\alpha$ determined by $R_1, \cdots, R_{n+1}$. Hence condition (7) implies that $R$ lies inside the solid angle $\alpha$. A corresponding statement with primes holds for $R'$. *Hence there exist* real constants $a_i$ and $a_i'$ such that

$$a_i \geqq 0, \qquad a_i' \geqq 0, \qquad \sum_{i=1}^{n+1} a_i \neq 0, \qquad \sum_{i=1}^{n+1} a_i' \neq 0,$$

and such that

$$R = a_i R_i, \qquad R' = a_i' R_i' \qquad (i \text{ summed}). \tag{10}$$

Multiplying (8) by $a_i a_j'$, summing on $i$ and $j$, one obtains

$$(a_i R_i) \cdot (a_j' R_j) \geqq (a_i R_i') \cdot (a_j' R_j'),$$

whence by (10)

$$R \cdot (a_j' R_j) \geqq (a_i R_i') \cdot R'. \tag{11}$$

Similarly multiplying (9) by $a_i$, summing on $i$, we get

$$R' \cdot (a_i R_i') > R \cdot R. \tag{12}$$

Conditions (11) and (12) imply that

$$R \cdot (a_j' R_j) > R \cdot R. \tag{13}$$

However multiplying (9) by $a_i'$, we get

$$R' \cdot R' = R' \cdot (a_i' R_i') > R \cdot (a_i' R_i). \tag{14}$$

Since $R \cdot R = R' \cdot R' = k^2$, conditions (13) and (14) are contradictory. Hence the assumption that $\Delta(x_1', \cdots, x_{n+1}')$ is not covered by the spheres $S_i'$ is *false*. Since $\Delta(x_1', \cdots, x_{n+1}') \cdot S_i' \cdot S_j' \neq 0$, and since $\Delta$ is covered by the spheres $S_i'$, a theorem[4] of Knaster, Kuratowski and

[4] See Alexandroff and Hopf [1, p. 377]. The theorem states: *If the closed sets $A_i$ cover the simplex $T$, and if each side $a_{i_1} \cdots a_{i_r}$ of $T$ is such that $a_{i_1} \cdots a_{i_r} \subset A_1 + \cdots + A_{i_r}$, then $A_1 \cdot A_2 \cdot \cdots \cdot A_{n+1} \neq 0$.*

Mazurkiewicz implies by induction that $\prod_{i=1}^{n+1} S_i' \neq 0$. Since condition (4) now holds for each set of $n+1$ of the spheres $S_i'$, the theorem of Helly[5] type implies that (4) holds for all the spheres $S_i'$.

We now readily prove the following corollary.

COROLLARY 1. *The property* E *holds for the n-dimensional hyperbolic space.*

For the hyperbolic space $M$ this corollary is an immediate consequence of the fact that $M$ can be defined as the points $(x_1, x_2, \cdots, x_{n+1})$ in the $(n+1)$-dimensional Euclidean space which are on one sheet of the hyperboloid[6]

$$k^2 x_1^2 - x_2^2 - x_3^2 - \cdots - x_{n+1}^2 = k^2.$$

Here $R_i \cdot R_j$ is defined to be the bilinear form

$$R_i \cdot R_j \equiv k^2 x_{i1} x_{j1} - x_{i2} x_{j2} - \cdots - x_{in+1} x_{jn+1},$$

and $F(u) = k \cosh^{-1}(u/k^2)$. These have the properties required for the proof of Theorem 1. A similar argument holds for the open hemispherical case.

The extensibility of $f(x)$ to the whole space $M$ so as to preserve condition (1) now follows as developed in the previous work of the author [3, pp. 105–106].

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[5] Loc. cit.

[6] Coxeter [5, pp. 209, 248].