ON THE GROWTH OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** In a recent paper by Boas, Boas and Levinson [1][1] two sets of sufficient conditions were given for the existence of $\lim_{x=\infty} y'(x)$ when $y(x)$ satisfies the differential equation

$$y'' + A(x)y = B(x). \tag{1:1}$$

We propose in this paper to use their methods and to generalize their results to the nth order linear differential equation

$$y^{(n)} + \sum_{i=1}^{n} A_i(x)y^{(n-i)} = B(x), \tag{1:2}$$

and to obtain sufficient conditions for

$$\lim_{x=\infty} y^{(n-1)}(x) \tag{1:3}$$

to exist. In case $n=2$, $A_1(x)=0$ and $A_2(x)=A(x)$, these conditions reduce to those in [1].

2. **Statements of the theorems.** In §4 we shall prove the following theorem.

THEOREM I. *If* $A_i(x)$ $(i=1, \cdots, n)$ *and* $B(x)$ *are continuous on* $0 \leqq x < \infty$, *and if the integrals*

$$\int_0^\infty x^{i-1} | A_i(x) | \, dx \qquad (i = 1, \cdots, n), \tag{2:1}$$

$$\int_0^\infty B(x)dx \tag{2:2}$$

exist, then the limit (1:3) *exists for any solution* $y(x)$ *of* (1:2).

We now write each function $A_i(x)$ as the difference of two nonnegative functions, $A_i(x) = A_i'(x) - A_i''(x)$, where $A_i' = (|A_i| + A_i)/2$, $A_i'' = (|A_i| - A_i)/2$. Then in §5, $\cdots$, §8 we shall prove the following theorem.

THEOREM II. *If* $A_i(x)$ $(i=1, \cdots, n)$ *and* $B(x)$ *are continuous on* $0 \leqq x < \infty$, *if the integrals*

Received by the editors October 30, 1943.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$$\int_0^\infty x^{i-1}A_i''(x)dx \qquad (i = 1, \cdots, n), \tag{2:3}$$

$$\int_0^\infty B(x)dx \tag{2:4}$$

exist, and if we have

$$\limsup_{x=\infty} x^{i-k-1}\int_x^\infty t^kA_i'(t)dt < 2(i-k-1)!k!/n(n-1) \tag{2:5}$$

whenever $i=2$, $k=0$, *or* $i=2j-1$, $2j$, $k=i-j, \cdots, i-2$ $(j=2, \cdots, [(n+1)/2])$, *where we agree that* $A_{n+1}'(t)=0$ *if* n *is odd, then the limit* (1:3) *exists for any solution* $y(x)$ *of* (1:2). *If in addition we have*

$$\int_0^\infty \sum_{i=1}^n x^{i-1}A_i'(x)dx = \infty, \tag{2:6}$$

then $\lim_{x=\infty} y^{(n-1)}(x)=0$.

3. **Some auxiliary lemmas.** In this section we state four lemmas needed in proving the main theorems. The first of these is found in [1].

LEMMA 1. *If* $f(x)$ *is continuous on* $0\leqq x<\infty$, *if* $M(x)$ *denotes the maximum of* $|f(t)|$ *on* $0\leqq t\leqq x$, *and if, for some positive numbers* α *and* x_0, $|f(x)|\leqq \alpha+M(x)/2$ $(x\geqq x_0)$, *then* $f(x)$ *is bounded on* $0\leqq x<\infty$.

LEMMA 2. *If* (2:5) *holds under the restrictions on* i *and* k *stated in Theorem* II, *then* (2:5) *also holds for* $i=2, \cdots, n$, $k=0, \cdots, i-2$.

This is manifestly true for $i=2$. If $i>2$ and if i is odd, then $i=2j-1$, $j\geqq 2$ and (2:5) holds if $k\geqq i-j=j-1$. Suppose now that $0\leqq k<j-1$. Then $i-k-1=2j-2-k>j-1$, and

$$x^{i-k-1}\int_x^\infty t^kA_i'(t)dt \leqq x^{j-1}\int_x^\infty t^{j-1}A_i'(t)dt,$$

$$\begin{aligned}\limsup_{x=\infty} x^{i-k-1}\int_x^\infty t^kA_i'(t)dt &< 2(j-1)!^2/n(n-1)\\ &< 2(2j-2-k)!k!/n(n-1).\end{aligned}$$

The reasoning when i is even is quite similar.

LEMMA 3. *If* $f(x)$ *is continuous and bounded on* $0\leqq x<\infty$, *and if the functions* $A_i(x)$ *satisfy the hypotheses of Theorem* I, *then all integrals of the form*

$$\int_0^\infty f(x)x^pA_i(x)dx, \qquad \int_0^\infty x^p\,|\,A_i(x)\,|\,dx \qquad (p = 0, \cdots, i-1)$$

exist. Under the hypotheses of Theorem II, *the same conclusion is valid provided that* $i=2, \cdots, n$, $p=0, \cdots, i-2$, *or that* $A_i(x)$ *be replaced by* $A_i''(x)$.

The proof of the first sentence of the lemma is immediate, and the second sentence follows similarly as soon as we refer to the preceding lemma.

LEMMA 4. *If* $y(x)$ *is of class* $C^{(m+q)}$ *on* $0 \leqq x < \infty$, *where* m *and* q *are non-negative integers, then*

$$q!\limsup_{x=\infty} x^{-q}\,|\,y^{(m)}(x)\,| \leqq \limsup_{x=\infty}\,|\,y^{(m+q)}(x)\,|. \tag{3:1}$$

To prove Lemma 4 we use Taylor's Theorem in the form

$$\begin{aligned} y^{(m)}(x) &= \sum_{k=0}^{q-1}(x-x_0)^k y^{(m+k)}(x_0)/k! \\ &\quad + \int_{x_0}^{x}\int_{x_0}^{t_{q-1}}\cdots\int_{x_0}^{t_1} y^{(m+q)}(t_0)dt_0dt_1\cdots dt_{q-1}. \end{aligned} \tag{3:2}$$

Let $M=\limsup_{x=\infty}\,|y^{(m+q)}(x)|$ and pick $e>0$. Then take x_0 so large that

$$|\,y^{(m+q)}(x)\,| < M + e \qquad (x \geqq x_0).$$

It follows from (3:2) that if $x \geqq x_0$

$$|\,y^{(m)}(x)\,| \leqq \sum_{k=0}^{q-1} x^k\,|\,y^{(m+k)}(x_0)\,|\,/k! + (M+e)x^q/q!,$$

$$q!\limsup_{x=\infty} x^{-q}\,|\,y^{(m)}(x)\,| \leqq M + e.$$

Since e is arbitrary, the statement of the lemma follows at once.

4. **Proof of Theorem** I. By virtue of (2:1) we can pick x_0 such that

$$\int_{x_0}^\infty \sum_{i=1}^{n} x^{i-1}\,|\,A_i(x)\,|\,/(i-1)!dx < 1/2. \tag{4:1}$$

If in (1:2) we substitute for $y^{(n-i)}$ $(i=2, \cdots, n)$ the values obtained from (3:2) by replacing m by $n-i$ and q by $i-1$, and solve the resulting equation for $y^{(n)}$, we get an equation which upon integration between the limits x_0 and x gives

$$
\begin{aligned}
y^{(n-1)}(x) = y^{(n-1)}(x_0) &- \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{x_0}^{x} A_i(t)\,\frac{(t-x_0)^k}{k!}\,y^{(n-i+k)}(x_0)dt \\
&- \sum_{i=1}^{n}\int_{x_0}^{x}\int_{x_0}^{t_{i-1}}\cdots\int_{x_0}^{t_1} A_i(t_{i-1})y^{(n-1)}(t_0)dt_0\cdots dt_{i-1} \\
&+ \int_{x_0}^{x} B(t)dt.
\end{aligned} \tag{4:2}
$$

Define the quantities B and α and the function $M(x)$ by the equations

$$
\begin{aligned}
B &= \max_{x_0 \leqq x \leqq \infty}\left|\int_{x_0}^{x} B(t)dt\right|, \\
\alpha &= \left| y^{(n-1)}(x_0)\right| \\
&\quad + \sum_{i=2}^{n}\int_{x_0}^{\infty}\left|A_i(t)\right|\sum_{k=0}^{i-2}\frac{t^k}{k!}\left| y^{(n-i+k)}(x_0)\right| dt + B, \\
M(x) &= \max_{0\leqq t \leqq x}\left| y^{(n-1)}(t)\right|.
\end{aligned} \tag{4:3}
$$

B exists by virtue of (2:2) and α exists by virtue of Lemma 3. From (4:2) and (4:1) we now get

$$
\left| y^{(n-1)}(x)\right| \leqq \alpha + M(x)/2 \qquad (x \geqq x_0). \tag{4:4}
$$

It follows from Lemma 1 that $y^{(n-1)}(x)$ is bounded on $0 \leqq x < \infty$. In this event we use Lemma 3 to see that the integrals involving $A_i(t)$ on the right side of (4:2) approach limits as $x \to \infty$. By (2:2) the integral of $B(x)$ approaches a limit. Therefore, $y^{(n-1)}(x)$ has a limit, proving Theorem I.

5. **Proof of Theorem II when $y^{(n-1)}(x)$ does not change sign for large values of x.** Then we may assume without loss of generality that x_0 is so large that $y^{(n-1)}(x) \geqq 0$ for $x \geqq x_0$ and that

$$
\int_{x_0}^{\infty}\sum_{i=1}^{n} x^{i-1}A_i''(x)/(i-1)!dx < 1/2. \tag{5:1}
$$

Since $y^{(n-1)}(t) \geqq 0$ on $t \geqq x_0$ and $-A_i(t) = A_i''(t) - A_i'(t) \leqq A_i''(t)$, we then have from (4:2) and (5:1) that (4:4) holds, the integrals in α existing by virtue of the second part of Lemma 3. It follows from Lemma 1 that $y^{(n-1)}(x)$ is bounded. Set $A_i = A_i' - A_i''$ in (4:2). Since $y^{(n-1)}(x)$ is bounded we see from Lemma 2 that all of the terms in (4:2) on the right side approach limits as $x \to \infty$ with the possible exception of

$$(5:2) \qquad -\sum_{i=1}^{n} \int_{x_0}^{x} \int_{x_0}^{t_{i-1}} \cdots \int_{x_0}^{t_1} A_i'(t_{i-1}) y^{(n-1)}(t_0) dt_0 \cdots dt_{i-1}.$$

Since $A_i' \geqq 0$, $y^{(n-1)}(t_0) \geqq 0$ for $t_0 \geqq x_0$, this term is a nonincreasing function of x which is bounded below since all the other terms in (4:2) are bounded. Hence it also approaches a limit. Therefore, the limit (1:3) exists.

6. **Proof of Theorem II when $y^{(n-1)}(x)$ changes sign infinitely many times.** Suppose first that $y^{(n-1)}(x)$ is bounded but that the limit (1:3) does not exist. Then we may assume without loss of generality that

$$\limsup \left| y^{(n-1)}(x) \right| = \limsup y^{(n-1)}(x) = M > 0.$$

Let x_m be a monotone sequence of points such that $x_m \to \infty$, $y^{(n-1)}(x_m) > 0$, $y^{(n-1)}(x_m) \to M$. Let a_m be the first point to the left of x_m such that $y^{(n-1)}(a_m) = 0$. We can suppose that a_1 is so large that for some $c < 1$ and for $i = 2, \cdots, n$, $k = 0, \cdots, i-2$ we have

$$(6:1) \qquad x^{i-1-k} \int_x^\infty t^k A_i'(t) dt < 2c(i-1-k)!k!/n(n-1) \qquad (x \geqq a_1).$$

By (4:2) with x_0 replaced by a_m and x replaced by x_m we have, if we observe that $y^{(n-1)}(t) \geqq 0$ on $a_m \leqq t \leqq x_m$,

$$(6:2) \qquad \begin{aligned} y^{(n-1)}(x_m) \leqq & \sum_{i=1}^{n} \int_{a_m}^{x_m} \int_{a_m}^{t_{i-1}} \cdots \int_{a_m}^{t_1} A_i''(t_{i-1}) y^{(n-1)}(t_0) dt_0 \cdots dt_{i-1} \\ & + \sum_{i=2}^{n} \sum_{k=0}^{i-2} \int_{a_m}^{x_m} \frac{t^k}{k!} A_i''(t) \left| y^{(n-i+k)}(a_m) \right| dt \\ & + \sum_{i=2}^{n} \sum_{k=0}^{i-2} \int_{a_m}^{x_m} \frac{t^k}{k!} A_i''(t) \left| y^{(n-i+k)}(a_m) \right| dt \\ & + \left| \int_{a_m}^{x_m} B(t) dt \right|. \end{aligned}$$

Since $y^{(n-1)}(t)$ is bounded, we see from Lemma 3 that the first sum on the right of (6:2) approaches zero as $m \to \infty$. By virtue of (2:4) so does the last term in (6:2). If we use Lemma 4 we discover that the upper limit of the third sum in (6:2) can not exceed

$$\begin{aligned} \limsup_{m=\infty} & \sum_{i=2}^{n} \sum_{k=0}^{i-2} \frac{\left| y^{(n-i+k)}(a_m) \right|}{k! a_m^{i-1-k}} \int_{a_m}^{x_m} t^{i-1} A_i''(t) dt \\ & \leqq \limsup_{x=\infty} \sum_{i=2}^{n} \sum_{k=0}^{i-2} \frac{\left| y^{(n-1)}(x) \right|}{k!(i-1-k)!} \int_x^\infty t^{i-1} A_i''(t) dt = 0. \end{aligned}$$

Finally we use (6:1) and Lemma 4 to see that the upper limit of the second sum in (6:2) cannot exceed

$$\begin{aligned}\limsup_{m=\infty} \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{a_m}^{\infty} \frac{t^k}{k!} A_m'(t) \left| y^{(n-i+k)}(a_m) \right| dt &\leqq \limsup_{x=\infty} \sum_{i=2}^{n}\sum_{k=0}^{i-2} \frac{2c(i-1-k)!\left| y^{(n-i+k)}(x)\right|}{n(n-1)x^{i-1-k}} \\ &\leqq \limsup_{x=\infty} \sum_{i=2}^{n}\sum_{k=0}^{i-2} 2c\left| y^{(n-1)}(x)\right| / n(n-1) = cM < M.\end{aligned}$$

Referring to (6:2) we see that we have reached a contradiction of our choice of the points x_m.

7. **Proof that $y^{(n-1)}(x)$ must be bounded.** To complete the proof of the first part of Theorem II, it is sufficient to prove that $y^{(n-1)}(x)$ must be bounded under the hypotheses (2:3), (2:4), (2:5) and the assumption that $y^{(n-1)}(x)$ changes sign infinitely many times. Suppose on the contrary that $y^{(n-1)}$ is unbounded. Then we can pick a sequence $x_m \to \infty$ such that

$$\left| y^{(n-1)}(x_m) \right| \geqq \left| y^{(n-1)}(x) \right| \qquad (x \leqq x_m), \tag{7:1}$$

$y^{(n-1)}(x_m)$ has the same sign, which we may suppose to be positive, and $y^{(n-1)}(x_m) \to \infty$. Let a_m be defined for x_m as in §6, and suppose that a_1 is so large that (6:1) holds. Using (7:1) and Taylor's Theorem (3:2) with m replaced by $n-i+k$, q replaced by $i-1-k$, x_0 replaced by 0, and x replaced by a_m, we find that

$$\left| y^{(n-i+k)}(a_m) \right| \leqq y^{(n-1)}(x_m) \frac{a_m^{i-k-1}}{(i-k-1)!} + \sum_{h=0}^{i-k-2} \left| y^{(n-i+k+n)}(0) \right| \frac{a_m^h}{h!}.$$

It now follows from (6:2) and (6:1) that

$$\begin{aligned} y^{(n-1)}(x_m) \leqq y^{(n-1)}(x_m) &\left\{ \sum_{i=1}^{n} \int_{a_m}^{\infty} \frac{t^{i-1}A_i''(t)}{(i-1)!} dt + c \right. \\ &\left. + \sum_{i=2}^{n}\sum_{k=0}^{i-2} \int_{a_m}^{\infty} \frac{t^{i-1}A_i''(t)}{k!(i-k-1)!} dt \right\} + \left| \int_{a_m}^{x_m} B(t)dt \right| \\ &+ \sum_{i=2}^{n}\sum_{k=0}^{i-2}\sum_{h=0}^{i-k-2} \frac{\left| y^{(n-i+k+h)}(0)\right|}{k!h!} \int_{a_m}^{\infty} t^{k+h}A_i(t)dt. \end{aligned}$$

Since all of the integrals on the right of this last inequality approach zero as $m \to \infty$ and $0 < c < 1$, we reach a contradiction.

8. **Proof of the second part of Theorem** II. Suppose on the contrary that $y^{(n-1)}(x)$ does not approach zero. Without loss of generality we may assume that

$$\lim_{x=\infty} y^{(n-1)}(x) = 2a > 0.$$

Then there exists an x_0 such that

$$3a > y^{(n-1)}(x) > a \qquad (x \geqq x_0). \tag{8:1}$$

Now set $A_i = A_i' - A_i''$ in (4:2) and let $x \to \infty$. Then all of the terms on the right approach limits with the possible exception of the term (5:2). Since $y^{(n-1)}(x)$ approaches a limit, so must (5:2). But by (8:1) we have

$$\int_{x_0}^{t_{i-1}} \cdots \int_{x_0}^{t_1} y^{(n-1)}(t_0)dt_0 \cdots dt_{i-2} > a(t_{i-1} - x_0)^{i-1}/(i-1)!.$$

Consequently, the term (5:2) is greater than

$$a\int_{x_0}^{x} \sum_{i=1}^{n} (t - x_0)^{i-1} A_i'(t)/(i-1)!dt.$$

By (2:6) and Lemma 3 this last integral becomes infinite as $x \to \infty$, so that (5:2) cannot approach a limit. This contradiction completes the proof of Theorem II.

Added in proof. Since the submission of this paper to the editors, it has come to the author's attention that Theorem I was proved by Otto Haupt, *Über das asymptotische Nerhalten der Lösungen gewisser linearer gewöhnlicher Differentialgleichungen*, Math. Zeit. vol. 48 (1942) pp. 282–292. Our proof, based on Lemma 1, seems distinctly simpler and certainly more elementary than that of Haupt. To the best of our present knowledge, Theorem II is new.

Bibliography

1. Boas, Boas and Levinson, *The growth of solutions of a differential equation*, Duke Math. J. vol. 9 (1942) pp. 847–853.

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