ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

Algebra and Theory of Numbers

1. A. A. Albert: *Quasigroups*. II.

A loop is a quasigroup with an identity element. In this part it is shown that H is a normal divisor of a loop G if and only if $(xH)(yH) \subset (xy)H$, $xH \subset (xH)h$, $(xy)H \subset x(yH)$ for every x and y of G and h of H. The intersection and union of two normal divisors of G are normal divisors of G, and the standard theorems used to prove the Jordan-Hölder theorem and the Schreier refinement theorem are valid for loops (although the proofs are very different). The paper shows how to extend the notion of solvable group to loops and also proves that various results of the theory of groups are also valid for loops. A construction is given of all loops with a given normal divisor and a given quotient loop, and the theory is applied to give an explicit determination of all loops of order six with a subloop of order three. Finally it is shown that all quasigroups of order five not isotopic to the group are isotopic to each other. (Received October 15, 1943.)

2. A. A. Albert: *Quasiquaternion algebras*

A quasiquaternion algebra has a basis $1, i, j, ji$ over a field F such that $i^2 = a + bi$, $ij = j(1-i)$, $j^2 = c$ for a, b, c in F, $c \neq 0$, $b \neq 1$. Also $b = 0$ when F has characteristic different from two and one writes $A = (a, c)$ in this case. All quadratic subalgebras are determined and it is shown that (a, c) is isomorphic to (a_0, c_0) if and only if $a_0 = a$, $c_0 = d^2c$ for $d \neq 0$ in F. The results in the characteristic two case are slightly different. A quasiquaternion algebra is a division algebra if and only if the algebras $F[i]$ and $F[j]$ are fields. If F is a finite field of q elements there are division algebras of this kind over F if and only if q is odd. Then there are $1/2(q-1)$ such algebras not isomorphic in pairs. The question as to when two algebras over F of characteristic not two are isotopic is completely solved for quasiquaternion division algebras. (Received October 15, 1943.)

3. J. L. Brenner: *The linear homogeneous group*. II.

Let (x_i) represent an n-tuple (vector) whose elements are residue classes mod p^r (p, prime; r, positive integer). The p^{nr} vectors (x_i) form a group $\mathfrak{A}$ under the operation vector addition: $(x_i)+(y_i) = (z_i)$, $z_i \equiv x_i + y_i \pmod{p^r}$. An automorphism of $\mathfrak{A}$ may be defined by specifying the n images (a_{ji}) of the generators $e_j = (0, \cdots, 1_j, \cdots, 0)$, where det $(a_{ji}) \neq 1 \pmod{p}$. $\mathfrak{G}_{p,n,r}$ is the group of these automorphisms. In this article the lattices of normal and of characteristic subgroups of $\mathfrak{G}$ are described; the lattices are distributive except when $n = p = 2$, in which case they are not distributive. $\mathfrak{N}_s$ consists of all matrices in $\mathfrak{G}$ which fix every element of order p^s in $\mathfrak{A}$. $\mathfrak{N}_s$ is characteristic in $\mathfrak{G}$. This article will appear in Ann. of Math. vol. 45 (January, 1944). (Received November 1, 1943.)

4. R. H. Bruck: *Some results in the theory of linear non-associative algebras.*

An algebra is defined to be isotopically simple if every isotope is simple. The theory of linear algebras is shown to be reducible to that of isotopically simple algebras. The following exhaustive classification of algebras is convenient: (I) all algebras with unit elements, and their isotopes; (II) all algebras with right units in which every element is a left-hand divisor of zero, and their isotopes; (III) all algebras anti-isomorphic to those of class II; (IV) all algebras in which every element is both a left-hand and right-hand divisor of zero. Isotopically simple algebras of order n over an arbitrary field are constructed as follows: of class I, for all $n \neq 2$; of classes II and III, for all $n > 3$. Isotopically simple algebras of order 2 are (isotopic to) quadratic fields. Certain types of Lie algebras (class IV) are shown to be isotopically simple. New simple algebras of all orders and new division algebras of orders 4, 8 and countable infinity are defined in terms of quasigroup algebras and generalized quasigroup algebras. The notion of generalized quasigroup algebra stems from the extension problem for quasigroups, of which a complete solution is given here for quasigroups of arbitrary order. (Received October 6, 1943.)

5. Claude Chevalley: *Some properties of ideals in rings of power series.*

The paper is concerned mainly with the behavior of prime ideals in power series rings under extension of the basic field. The theorem of A. Weil on the existence of the field of definition of a prime ideal in the ring of polynomials is generalized for prime ideals in rings of power series. (Received October 8, 1943.)

6. M. M. Day: *Arithmetic of ordered systems.*

Operations of ordered addition and ordered multiplication of (partially) ordered systems are defined to include as special cases the ordinal and cardinal sum and product and the ordinal exponentiation of G. Birkhoff (Duke Math. J. vol. 9 (1942) pp. 283–302). If $(T, \geqq)$ and $(S_t, \geqq)$ are ordered systems, the ordered product over $(T, \geqq)$ of the $(S_t, \geqq)$ has as elements the functions f defined on T with $f(t)$ in S_t for every t: $f \geqq' f_1$ means that if $f(t) \neq f_1(t)$ there exists $t_1 \geqq t$ such that $f(t_1) > f_1(t_1)$ (essentially lexicographic ordering); the relation $\geqq$ in the ordered product is the least transitive relation including $\geqq'$. The product over the system of integers of a family of two-element well-ordered systems is a simple example in which $\geqq'$ is not transitive. The principal results depend on the fundamental theorem: If $(T, \geqq)$ is a number and each $(S_t, \geqq)$ is transitive, the relation $\geqq'$ is almost transitive; that is if $f_1 \geqq' f_2 \geqq' f_3 \geqq' \cdots \geqq' f_n$, there exists f_0 such that $f_1 \geqq' f_0 \geqq' f_n$. (Received October 22, 1943.)

7. R. P. Dilworth: *A decomposition theorem for partially ordered sets.*

Let P be a partially ordered set. P has finite *width* k if k is the least integer such that every set of $k+1$ elements contains a comparable pair. It is shown that *if P has finite width k, then P is the set sum of k disjoint chains.* An almost trivial application yields the P. Hall theorem on representatives of sets. Indeed, the above theorem seems to be the natural infinite extension of the P. Hall theorem. Application to distributive lattices gives the following result: *Let* $m = \max h(a)$ *where* $h(a)$ *is the number of elements covering an element* a *of a distributive lattice* L. *Then* L *is a sublattice of a direct product of* m *chains and* m *is the least number of chains for which this holds.* (Received October 22, 1943.)

8. Roy Dubisch and Sam Perlis: *On the radical of a non-associative algebra.* (Preliminary report.)

The radical of an associative algebra may be defined in many equivalent ways. In the case of a non-associative algebra most of these definitions provide subsets which are not always ideals or which lack many of the common properties of the radicals of associative algebras. One of these common properties is preserved by the definition made recently by Albert for non-associative algebras obeying a simple hypothesis, but most of the other properties are lost. It is the purpose of this paper to investigate several definitions yielding ideals with at least one of the desired, well known attributes of a radical, to see if any two of these ideals coincide or satisfy an inclusion relation and to see if any one of them has some properties of particular interest. The paper begins with the special case of alternative algebras by giving a new and simple proof of Zorn's theorem that the totality R of properly nilpotent elements forms an ideal. The method consists of showing that the definition given in 1941 by one of the authors provides an ideal H and that $H = R$. Albert's radical N is shown to coincide with H and R for this case. But there are examples of non-alternative algebras for which the property $H = R = N$ fails. (Received October 29, 1943.)

9. Samuel Eilenberg and Saunders MacLane: *Cohomology theory in groups.*

Let π be a discrete group which acts as a group of left operators on an abelian group G. A function f of n variables in π and with values in G is called an n-cochain. The coboundary δf of f is an $(n+1)$-cochain defined by setting $\delta f(x_1, \cdots, x_{n+1}) = x_1[f(x_2, \cdots, x_{n+1})] - f(x_1x_2, \cdots, x_{n+1}) + f(x_1, x_2x_3, \cdots, x_{n+1}) \cdots \pm f(x_1, \cdots, x_{n-1}, x_nx_{n+1}) \mp f(x_1, \cdots, x_n)$. Since $\delta\delta = 0$, the customary construction yields a cohomology group $H_n(\pi, G)$ of π with coefficients in G. By a suitable definition of a cup product, the groups $H_n(\pi, G)$ give rise to a cohomology ring. The group $H_1(\pi, G)$ is the group of crossed homomorphisms of π in G (functions such that $f(x_1x_2) = f(x_1) + x_1f(x_2)$) modulo the principal homomorphisms (functions $f(x) = x(g) - g$, for a constant $g \in G$). The group $H_2(\pi, G)$ is the group of group extensions of G by π, with the indicated operators. If π is represented as a factor group F/R of a free group F, then $H_n(\pi, G) = H_{n-2}(\pi, \text{Hom}(R, G))$, with the group π suitably operating on the group Hom (R, G) of homomorphisms of R into G. (Received November 5, 1943.)

10. Wade Ellis: *Relations satisfied by linear operators on a vector space.*

Let α, β be two linear operators on a linear vector space V of dimension n. If $n = 2$, all general relations on α, β are consequences of the three known relations, of which two involve only one operator each (Cayley), and the third expresses $\alpha\beta + \beta\alpha$ in terms of α, β, 1 (Laguerre). If $n > 2$, there are, in addition to those of a similar type, other relations (with invariants of certain kinds as coefficients) which it is the purpose of this paper to find and express. When $n = 3$, an important distinction is made between the case (I) (of simultaneous symmetrizability) where there is a coordinate system in which α, β are both represented by symmetric matrices, and the case (II) where this is not true. In I the relations are expressed as determinants of order seven, with all their elements rational invariants. In II the coefficients must involve a quadratic irrationality δ; it is convenient to use determinants of order nine whose elements involve δ and rational invariants. For $\delta=0$, these relations reduce to those of case I. Special cases of the relations occur in connection with the study of certain differential equations of physics. (Received October 23, 1943.)

11. N. J. Fine and Ivan Niven: *The probability that a determinant be congruent to a* (mod *m*).

A complete solution is given to the problem of evaluating $P_n(a, m)$, the probability that a determinant of order n having integral elements be congruent to a modulo m. (Received October 15, 1943.)

12. A. L. Foster and B. A. Bernstein: *Symmetric definition and duality theorem for rings.*

It is shown that commutative rings with unit are capable of symmetric definition and possess an elementary principle of duality of which the Boolean algebra duality is an instance. In this paper only non-ideal theoretic consequences of this duality are treated, and applications are made to the special case of fields. (Received November 29, 1943.)

13. Ralph Hull: *A theorem on the unit groups of simple algebras.*

Let $\mathfrak{A}$ be a normal simple algebra of degree n over an algebraic number field $\mathfrak{F}$. If $\mathfrak{A}$ satisfies condition R, that is, $n>2$ or $\mathfrak{A}$ is unramified at at least one infinite prime place of $\mathfrak{F}$ when $n=2$, then any two distinct maximal orders of $\mathfrak{A}$ have distinct unit groups. The proof is based on the splitting-field theory of algebras $\mathfrak{A}$ which satisfy R (Eichler, Math. Zeit. vol. 43 (1938) pp. 481–494) and local arithmetical considerations. Condition R is indispensable in general. The theorem can be used to describe the Brandt groupoid of normal ideals of $\mathfrak{A}$. (Received October 18, 1943.)

14. Jakob Levitzki: *A characteristic minimal condition for semi-primary rings.*

A ring S is called semi-primary if the sum R of all two-sided nilpotent ideals of S is nilpotent and S/R is semi-simple. It has been proved by Hopkins (C. Hopkins, *Nil-rings with minimal condition for admissible left ideals*, Duke Math. J. vol. 4 (1938) pp. 664–667) that a ring S with minimal condition for left-ideals is semi-primary. But neither this condition, nor the weaker assumptions found later by other authors are necessarily satisfied by each semi-primary ring. In the present note it is shown that the following minimal condition is necessary as well as sufficient for semi-primary rings: Each descending chain of the form $L_1 \supset L_2 \supset L_3 \supset \cdots$, where the L_i are left-ideals containing R, and each descending chain of the form $A_1 \supset A_1 \cdot A_2 \supset A_1 \cdot A_2 \cdot A_3 \supset \cdots$, where the A_i are two-sided ideals contained in R, is finite. (Received October 27, 1943.)

15. G. Y. Rainich: *Noncommutative relations.*

The problem of factorization of polynomials in a ring discussed previously (abstract 48-1-51) leads to the consideration of systems of relations (in terms of multiplication and addition) on indeterminates (which first appear as indeterminate coefficients). The situation is studied from several points of view. I (Abstract point of view): One set of relations may imply another. A set may be contradictory. Examples are given of complete sets; a set is complete when any relation compatible with it is implied by it. II (Realization): Here a linear vector space is considered and linear operators on it which satisfy the same relations as those that are given. The connection with I is given by the fact that relations satisfied in an invariant subspace imply relations in the whole space. In III a ring is considered generated (with the aid of a field) by operators satisfying given relations. The special case when the relations involve multiplication only correspond to a group algebra. IV deals with relations satisfied by operators as a result of their being operators on a vector space of a given dimensionality. (Received October 23, 1943.)

16. H. E. Salzer: *New tables and facts involving sums of four tetrahedral numbers.*

The author has a second empirical theorem about tetrahedral numbers, that is, $(n^3-n)/6$ for integral n. Every tetrahedral number greater than 1 is the sum of four other non-negative tetrahedrals. This theorem has been verified for the first 200 cases in a table expressing every tetrahedral from 4 through 1373701 as a sum of four non-negative tetrahedrals. With the exception of 153, the first 200 triangular numbers $n(n+1)/2$ can each be expressed as the sum of four non-negative tetrahedrals. There are only 45 integers less than or equal to 1000 which cannot be expressed as the sum of four non-negative tetrahedrals. All numbers ending in 0, 5, or 6 which are less than or equal to 2006 are expressible as a sum of four non-negative tetrahedrals. This includes the first 201 cases of each type. It is interesting to note that the smallest example of a number ending in 4 which is not expressible as a sum of four non-negative tetrahedrals occurs at 1314. Thus here is an instance where a statement is true in the first 131 cases, but fails in the 132nd. (Received October 13, 1943.)

17. L. R. Wilcox: *Modularity in Birkhoff lattices.*

The following theorem connecting G. Birkhoff's upper semi-modular lattices with the author's M-symmetric lattices is proved. Let a lattice be called upper semi-modular if $a+b$ covers a, b when a and b cover ab; let a lattice be called M-symmetric if $(a+b)c=a+bc$ for every $a\leqq c$ implies $(d+c)b=d+cb$ for every $d\leqq b$; finally, let a lattice be called of finite dimensional type if every a, b with $a<b$ have a finite principal chain connecting them. Then a lattice of finite dimensional type is upper semi-modular if and only if it is M-symmetric. The purpose of this theorem is to replace the condition of Birkhoff, forceful only when some chain condition is assumed, by a strictly algebraic condition which is suitable for use in the infinite dimensional case. (Received October 19, 1943.)

Analysis

18. Stefan Bergman: *The determination of singularities of functions satisfying a partial differential equation from the coefficients of their series development.*

Let $U(z, \bar{z})=A_{00}+\sum_{m=0}^{\infty}\sum_{m=1}^{\infty}A_{mn}z^m\bar{z}^n$ be a (complex) solution of the equation $L(U)\equiv U_{z\bar{z}}+a_1U_z+a_2U_{\bar{z}}+a_3U=0$ where a_k, $k=1, 2, 3$, are entire functions of two variables $z=x+iy$, $\bar{z}=x-iy$, x, y real. Using the results of the papers Rec. Math.