remarked that Theorem A may well carry, in such a study, a weight greater than that indicated by its relatively minor role in the proof of Theorem B.

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THE EQUIVALENCE OF n-MEASURE AND LEBESGUE MEASURE IN E_n

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Consider a set A of points in euclidean n-space E_n. For each countable covering $\{A_i\}$ of A by arbitrary sets consider the sum

$$\sigma = \sum_i c_m \delta(A_i)^m,$$

where m is a fixed positive number, $c_m = \pi^{m/2}/2^m\Gamma[(m+2)/2]$, and $\delta(A)$ is the diameter of A. The constant c_m is, for integral m, the m-volume of a sphere of unit diameter in E_m. Let $L_m(A;\alpha)$ be the greatest lower bound of all sums σ corresponding to coverings for which $\delta(A_i) < \alpha$ for all i $(\alpha > 0)$. We define the m-measure of A as $L_m(A) = \lim_{\alpha \to 0} L_m(A;\alpha)$. We denote the outer Lebesgue measure of A by $|A|$.

We shall show that *n-measure and outer Lebesgue measure are equal*: $L_n(A) = |A|$. A statement on this matter by W. Hurewicz and H. Wallman is true but misleading: these authors assert that $L_n(A)/c_n$ and $|A|$ may be unequal.[1]

F. Hausdorff has introduced an m-measure $L_m^S(A)$ defined as is $L_m(A)$ except that coverings by spheres are used instead of coverings by arbitrary sets. He has shown[2] that $L_n^S(A) = |A|$. However $L_m(A)$ and $L_m^S(A)$ are unequal in general, as A. S. Besicovitch has shown[3] for $m=1$, $n=2$. S. Saks[4] and others define m-measure as $L_m(A)/c_m$.

Our proof, which is an obvious extension of Hausdorff's proof, depends on two known theorems.

THEOREM I. *Of all sets in E_n having a given diameter, the n-sphere has the greatest outer Lebesgue measure.*[5]

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[1] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941, p. 104.

[2] F. Hausdorff, *Dimension und äusseres Mass*, Math. Ann. vol. 79 (1919) p. 163.

[3] A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points*, Math. Ann. vol. 98 (1928) pp. 458–464. R. L. Jeffery, *Sets of k-extent in n-dimensional space*, Trans. Amer. Math. Soc. vol. 35 (1933) p. 634.

[4] S. Saks, *Theory of the integral*, Warsaw, 1937, pp. 53–54.

THEOREM II. *Suppose that to each point x of a set A in E_n there corresponds a set of closed n-spheres centered at x of arbitrarily small positive diameter. Then for any given $\epsilon > 0$, a countable number of the spheres cover A and are such that the sum of their Lebesgue measures is at most $|A| + \epsilon$.*[6]

We now prove that

$$|A| \leqq L_n(A) \leqq L_n^S(A) \leqq |A|.$$

For any countable covering $\{A_i\}$ of A,

$$|A| \leqq \sum_i |A_i| \leqq \sum_i c_n \delta(A_i)^n$$

by Theorem I. Hence $|A| \leqq L_n(A; \alpha)$ for all α and $|A| \leqq L_n(A)$.

The definitions imply that $L_n(A) \leqq L_n^S(A)$.

Finally, given $\epsilon > 0$ and $\alpha > 0$, assign to each point x of A the set of all closed spheres centered at x and of positive diameter less than α. Then by Theorem II a countable number of these spheres $\{S_i\}$ cover A and are such that

$$\sum_i |S_i| = \sum_i c_n \delta(S_i)^n \leqq |A| + \epsilon.$$

Hence $L_n^S(A; \alpha) \leqq |A|$ and $L_n^S(A) \leqq |A|$.

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[5] W. H. and G. C. Young, *The theory of sets of points*, Cambridge, 1906, pp. 293–294. L. Bieberbach, *Über eine Extremaleigenschaft des Kreises*, Jber. Deutschen Math. Verein. vol. 24 (1915) pp. 247–250. T. Kubota, *Über die konvex-geschlossenen Mannigfaltigkeiten im n-dimensionalen Räume*, Science Reports, Tôhoku Imperial University, vol. 14 (1925) p. 98. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Ergebnisse der Mathematik vol. 3 (1934) pp. 76 and 107. W. Feller, *Some geometric inequalities*, Duke Math. J. vol. 9 (1942) pp. 889–892. The diameter of an arbitrary set B equals the diameter of the smallest closed convex set containing B.

[6] H. Rademacher, *Eineindeutige Abbildung und Messbarkeit*, Monatshefte für Mathematik und Physik vol. 27 (1916) p. 190. The case $|A| = \infty$ is not excluded.