REFERENCES

1. Zygmund, *Trigonometrical series*, chap. 5, p. 123.
2. Ibid., chap. 2, p. 32.

UNIVERSITY OF WISCONSIN

ON FIBRE SPACES. II

RALPH H. FOX

This paper is primarily concerned with fibre mappings[1] into an absolute neighborhood retract. Theorem[2] 3 is a converse of the covering homotopy theorem; it characterizes fibre mappings (into a compact ANR) as mappings for which the covering homotopy theorem holds. Theorem 4 is Borsuk's fibre theorem;[3] the proof[4] which I present here is new. It seems to me that this theorem is a promising tool in function-space theory. Also I think that it furnishes conclusive justification for the generality of the Hurewicz-Steenrod definition of a fibre space. In fact, a fibre space of the type constructed by Borsuk's theorem almost never has a compact base space and almost never has its fibres of the same topological type.

The common denominator of the proofs of Theorems 3 and 4 is a property which I call *local equiconnectivity*. Local equiconnectivity is a strengthened form of local contractibility and a weakened form of the absolute neighborhood retract property (Theorems 1 and 2). Definitions and notations are those of FS. I.[5]

Let Δ be the diagonal subset $\sum_{b\in B}(b, b)$ of $B\times B$. I shall call the space B *locally equiconnected* (or, to be specific, (U, V)-equiconnected) if there are neighborhoods U and V of Δ and a homotopy λ in B between the two projections of U which does not move the points of Δ and which is uniform[5] with respect to V. Precisely:

(1) $\lambda_t(b_0, b_1)$ is defined for all $(b_0, b_1)\in U$,

(2) $\lambda_0(b_0, b_1) = b_0$,

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[1] W. Hurewicz and N. Steenrod, Proc. Nat. Acad. Sci. U. S. A. vol. 27 (1941) p. 61.

[2] This theorem was announced in Hurewicz-Steenrod, op. cit. footnote 3.

[3] K. Borsuk, Fund. Math. vol. 28 (1937) p. 99.

[4] This proof was announced in the author's paper *On the deformation retraction of some function spaces* · · · , Ann. of Math. vol. 44 (1943) p. 52.

[5] $\bar{\pi}(x, b) = (\pi(x), b)$ as in R. H. Fox, *On fibre spaces*. I, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 555–557.

(3) $\lambda_1(b_0, b_1) = b_1$,

(4) $\lambda_t(b, b) = b$ for every $(b, b) \in \Delta$, $0 \leqq t \leqq 1$,

(5) there is a $\delta > 0$ such that $|t - t'| < \delta$ implies that $\sum_{(b_0, b_1) \in U} (\lambda_t(b_0, b_1), \lambda_{t'}(b_0, b_1)) \subset V$.

Roughly speaking, B is locally equiconnected if there are paths between sufficiently nearby points such that the paths depend continuously on the end points.

THEOREM 1. *A locally equiconnected space is locally contractible.*

Let N be a neighborhood of some point b_1 of B and let M denote the set of points b_0 such that $\sum_{0 \leqq t \leqq 1} \lambda_t(b_0, b_1) \subset N$. By (4), $b_1 \in M$; a simple continuity argument shows that M is a neighborhood of b_1. Since M is contractible to b_1 in N the theorem is proved.

THEOREM 2. *A compact ANR-set is locally equiconnected.*

Let B be a neighborhood retract of the Hilbert parallelotope Q and let r be a retraction of an open neighborhood N of B onto B. Since $Q - N$ and B are disjoint compact sets $\epsilon = d(B, Q - N)/2 > 0$. Let U_ϵ be the closed neighborhood of Δ determined by the covering of B by ϵ-spheres and let $\lambda_t(b_0, b_1) = r((1-t)b_0 + tb_1)$ for $(b_0, b_1) \in U_\epsilon$, $0 \leqq t \leqq 1$. Conditions (1), (2), (3), and (4) are obviously satisfied. Condition (5) follows, for any V, from the compactness of U_ϵ.

From Theorems 1 and 2 it follows,[6] for finite dimensional compacta, that local contractibility, local equiconnectivity and the ANR property are equivalent. For infinite dimensional spaces no more is known than is implied above.

THEOREM 3 (CONVERSE OF THE COVERING HOMOTOPY THEOREM). *Let B be a (U, V)-equiconnected space and let $\pi \in B^X$. Suppose that for every mapping $g \in X^Y$ and homotopy h in B which is uniform with respect to V and has initial value[5] πg there exists a covering homotopy h^* in X with initial value g. Then π is a fibre mapping relative to U.*

Let $h_t(x, b) = \lambda_t(\pi(x), b)$. Since h is uniform with respect to V there is a covering homotopy h^* such that $h_0^*(x, b) = x$. Let $\phi(x, b) = h_1^*(x, b)$. Then[5] ϕ maps $\bar{\pi}^{-1}(U)$ continuously into X and $\pi\phi(x, b) = b$. Since $h_{[0,1]}(x, \pi(x)) = \pi(x)$ it follows that $\phi(x, \pi(x)) = h_1^*(x, \pi(x)) = h_0^*(x, \pi(x)) = x$. Thus ϕ is a slicing function.

Let A be a closed subset of X and let π denote the sectioning operation $\pi(f) = f | A$, $f \in Y^X$.

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[6] K. Borsuk, Fund. Math. vol. 19 (1932) p. 240, Theorem 32.

THEOREM 4 (BORSUK'S FIBRE THEOREM). *If A is closed in X and Y is a compact ANR-set then π is a fibre mapping.*

By Theorem 2, Y is locally equiconnected and, if it is suitably metrized, there is a positive number ϵ such that $\lambda_t(y_0, y_1)$ is defined whenever $d(y_0, y_1) < \epsilon$. Let Γ_0 denote the graph of π and let Γ_ϵ denote the subset of $Y^X \times Y^A$ defined by the rule $(f, g) \in \Gamma_\epsilon$ when $d(\pi(f), g) < \epsilon$. Because Y is compact Γ_ϵ is a neighborhood of Γ_0. Define

$$\psi_t(f, g, x) = \begin{cases} \lambda_t(f(x), g(x)) & \text{for} \quad (f, g, x) \in \Gamma_\epsilon \times A, \\ f(x) & \text{for} \quad (f, g, x) \in \Gamma_0 \times X. \end{cases}$$

Thus ψ is a homotopy in Y; each ψ_t is defined on the closed subset $C = \Gamma_\epsilon \times A + \Gamma_0 \times X$ of $\Gamma_\epsilon \times A$. But $\psi_0(f, g, x) = f(x)$ for every $(f, g, x) \in C$, and this map has the extension $\psi_0^*(f, g, x) = f(x)$ defined for every $(f, g, x) \in \Gamma_\epsilon \times X$. It follows[7] that ψ_1 can be extended to $\Gamma_\epsilon \times X$. Let ψ_1^* denote an extension of ψ_1 and set $\phi(f, g)(x) = \psi_1^*(f, g, x)$ for $(f, g) \in \Gamma_\epsilon$ and $x \in X$, so that $\phi(f, g) \in Y^X$ for every fixed $(f, g) \in \Gamma_\epsilon$. Then ϕ maps Γ_ϵ into Y^X, $\pi\phi(f, g) = g$, $\phi(f, \pi(f)) = f$. Thus ϕ is a slicing function for π.

Since the image set of a fibre mapping is necessarily open and closed in the base space, an example[8] "$\mathcal{E}$" shows that Theorem 4 is false for non-compact ANR-sets Y. However if neither X nor Y are compact (as in "$\mathcal{E}$") the topology of Y^X (and also of Y^A) depends on the metrization of Y. Thus it may be possible (as it is in "$\mathcal{E}$") to remetrize an ANR-set Y so as to make the sectioning operations fibre mappings. It should be observed that Borsuk has shown that Theorem 4 is false (with or without remetrization) if Y is not locally contractible.[4]

UNIVERSITY OF ILLINOIS

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[7] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941, p. 86.

[8] R. H. Fox, Bull. Amer. Math. Soc. vol. 48 (1942) p. 271 footnote 3.