EXACT nTH DERIVATIVES

HOWARD LEVI

Let y be a function of x with derivatives of all orders, and let θ be a function of x, y, and a finite number of derivatives of y. If, independently of the choice of the function y, θ is the nth total derivative of some function ψ of x, y, and derivatives of y, then we shall call θ an *exact nth derivative*. The problem with which this note is concerned is to determine, for any given function θ and positive integer n, if θ is an exact nth derivative. The case for which $n=1$ is completely covered by the well known Euler differential equation which arises in the simplest problem of the calculus of variations. For a function θ to be an exact first derivative, it is necessary and sufficient that θ satisfy the Euler differential equation. The contribution of this paper is the treatment of the cases in which n exceeds unity. A system of n differential equations is developed, satisfaction of which by θ constitutes a necessary condition that θ be an exact nth derivative. These equations do not yield an altogether satisfactory sufficient condition. It turns out that if θ satisfies the equations in question, it may still fail to be an exact nth derivative. However, under these circumstances, θ must differ from an exact nth derivative by a function of very special character.

Notation. Let us suppose y to be an arbitrary function of x possessing derivatives of all orders. We shall denote the jth derivative of y with respect to x by y_j, and sometimes denote y itself by y_0. We suppose θ to be a function of x, y, and of finitely many of the y_j, possessing partial derivatives of all orders with respect to all its arguments. The operation of differentiation with respect to x will be indicated by the symbol D; thus $D=\partial/\partial x+\sum y_{i+1}\partial/\partial y_i$. We shall understand that the range of the subscript i in D extends from zero to plus infinity, recognizing that when D operates on a function of x, y, and of finitely many of the y_j it reduces to a finite sum. The symbol D^i, where i is a positive integer, will denote the operation of taking the ith derivative. We shall use the expression $C_{p,q}$ to denote the binomial coefficient $p\cdot(p-1)\cdots(p-q+1)/q!$ where q is a nonnegative integer and p is any integer.

Summary of results. Let n be a positive integer. Let operators E_t, $t=1,\cdots,n$, be defined as follows. Expand, formally, $E_t=(1+D\partial/\partial y_1)^{-t}\partial/\partial y$ as the product by $\partial/\partial y$ of a power series in $D\partial/\partial y_1$, and replace terms $(D\partial/\partial y_1)^i\partial/\partial y$ by $D^i\partial/\partial y_i$. *Let there be a*

Received by the editors January 15, 1943.

function ψ *of* x, y, *and of finitely many* y_j *such that* $D^n\psi = \theta$ *identically in* x *and* y. *Then*[1]

$$E_t\theta = 0, \qquad t = 1, 2, \cdots, n, \tag{1}$$

identically in x *and* y.

We also establish a partial converse to this result. *Let* θ *satisfy equations* (1) *identically in* x *and* y. *Then there exist polynomials in* x $\pi_1, \pi_2, \cdots, \pi_{n-1}$, *where the degree of each nonzero* π_i *is less than* i, *such that* $\theta - \pi_1 y_1 - \cdots - \pi_{n-1}y_{n-1}$ *is an exact nth derivative.* Note that no term in y_0 need be subtracted from θ to render θ exact, so that for n equal to unity we see that θ itself is a first derivative if it satisfies (1). The equation $E_1\theta = 0$ is simply the Euler differential equation, so that for n equal to unity our result restates the well known property of the Euler equation.

Necessity proof. We use the relations

$$\frac{\partial}{\partial y_r} D^n = \sum_{i=0}^{\min(r,n)} C_{n,i} D^{n-i} \frac{\partial}{\partial y_{r-i}}, \tag{2}$$

$$r = 0, 1, 2, \cdots; n = 1, 2, 3, \cdots,$$

which we proceed to establish. For r equal to zero, this amounts to the statement that $(\partial/\partial y)D^n = D^n \partial/\partial y$, $n = 1, 2, 3, \cdots$. Clearly

$$\frac{\partial}{\partial y} D = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} + \sum y_{i+1}\frac{\partial}{\partial y_i}\right) = \frac{\partial^2}{\partial y \partial x} + \sum y_{i+1}\frac{\partial^2}{\partial y \partial y_i}.$$

Since the order in which the partial derivatives are extracted can be reversed, this last expression equals $D\partial/\partial y$. This disposes of the case $r = 0$, $n = 1$, and the cases $r = 0$, n arbitrary, follow immediately.

We now treat the cases in which r is greater than zero. In evaluating $(\partial/\partial y_r)D$ we must consider that y_r appears in the term $y_r \partial/\partial y_{r-1}$ as a coefficient of a partial derivative, which was not the case for r equal to zero. We obtain then

$$\frac{\partial}{\partial y_r} D = D\frac{\partial}{\partial y_r} + \frac{\partial}{\partial y_{r-1}}.$$

Consequently,

$$\frac{\partial}{\partial y_r} D^n = D\frac{\partial}{\partial y_r} D^{n-1} + \frac{\partial}{\partial y_{r-1}} D^{n-1}.$$

Proceeding by induction, we assume that the right member of this

[1] These equations are written out explicitly as equations (4) below.

equation has a representation

$$D\sum_{i=0}^{\min(r,n-1)} C_{n-1,i}D^{n-1-i}\frac{\partial}{\partial y_{r-i}}+\sum_{i=0}^{\min(r-1,n-1)} C_{n-1,i}D^{n-1-i}\frac{\partial}{\partial y_{r-1-i}}.$$

Collecting similar terms, we see that the coefficient of $D^{n-i}\partial/\partial y_{r-i}$ is $C_{n-1,i}+C_{n-1,i-1}$. It is well known that this sum is $C_{n,i}$. This suffices to prove that if relations (2) hold for values less than n and less than or equal to r, they likewise hold for n and r. It remains to discuss the transition from n and $r-1$ to n and r. This part of the proof is so similar to the foregoing discussion that the details may be omitted.

From now on n is to be a positive integer, arbitrary but fixed.

We multiply both members of the rth of equations (2) by D^r, $r=1, 2, \cdots$. As a consequence, the power of D which multiplies any given $\partial/\partial y_s$ is the same in all the modified equations (2) in which the expression appears, namely the $(n+s)$th power.

Denote the terms $D^r(\partial/\partial y_r)D^n$ by Λ_r, $r=0, 1, 2, \cdots$. After the above multiplication, equations (2) furnish us expressions for the Λ_r as linear combinations, with numerical coefficients, of terms $D^{n+s}\partial/\partial y_s$. The numerical coefficients can be readily obtained. A term $D^{n+s}\partial/\partial y_s$ appears first in the expression for Λ_s, and is present, in all, in $\Lambda_s, \Lambda_{s+1}, \cdots, \Lambda_{s+n}$. Its numerical coefficient in each of these Λ_{s+i} is $C_{n,i}$.

We now seek numbers $a_0, a_1, a_2, \cdots$, such that $\sum_{i=0}^{\infty}a_i\Lambda_i$ is identically zero; that is, in this sum the numerical coefficient of each term $D^{n+s}\partial/\partial y_s$ is to be zero. This implies that the a_i must satisfy

$$\sum_{i=0}^{n} a_{r+i}C_{n,i}=0, \qquad r=0, 1, 2, \cdots. \tag{3}$$

In order to solve these equations with a minimum of computation, we formulate an equivalent problem involving power series. Let u be an indeterminate and let $\sum a_i u^i$ be a formal power series in u, with numerical coefficients. Observe that in the product $(1+u)^n\sum a_i u^i$ the coefficient of u^{n+r}, $r=0, 1, 2, \cdots$, is precisely the left member of the corresponding equation (3). Thus if we can attribute values to the a_i which make the above product a polynomial of degree less than n, these a_i will also be solutions of (3). But the solution of this problem is immediate. Expand any one of the fractions $1/(1+u)^t$, $t=1, \cdots, n$, as a power series in u. The coefficients obtained in this way obviously serve as solutions for the polynomial problem, and, consequently, furnish solutions for equations (3).

The proof can now be quickly completed. Let θ be the nth derivative of a function ψ; in symbols $\theta = D^n\psi$. Then $D^r\partial\theta/\partial y_r = \Lambda_r\psi$. Since $\sum a_r\Lambda_r = 0$ we have $\sum a_r D^r \partial\theta/\partial y_r = 0$. Using explicit values for the a_r we have

$$\frac{\partial\theta}{\partial y} + \sum_{r=1}^{\infty} C_{-t,r} D^r \frac{\partial\theta}{\partial y_r} = 0, \qquad t = 1, 2, \cdots, n. \tag{4}$$

The details of the proof make plain the connection between equations (4) and equations (1).

Sufficiency proof. We need the following lemma.

Let n be a positive integer and let $\pi(u)$ be a polynomial in u, not identically zero, whose degree is less than n. Then in the expansion of the function $\mu(u) = \pi(u)/(1+u)^n$ as a power series in u the coefficients of no n consecutive powers of u are zero.

This is obviously true for n equal to unity, for the series of $1/(1+u)$ has no missing terms, and multiplication by a nonzero constant cannot introduce any such terms. We prove the lemma by induction on n. Representing $\pi(u)$ as a polynomial in $(1+u)$, we have $\pi(u) = b_0 + b_1(1+u) + \cdots + b_{n-2}(1+u)^{n-2} + b_{n-1}(1+u)^{n-1}$ where not all the b_i are zero. After being multiplied by $(1+u)$ our function has the form

$$(1+u)\mu(u) = [b_0 + b_1(1+u) + \cdots + b_{n-2}(1+u)^{n-2}]/(1+u)^{n-1} + b_{n-1}. \tag{5}$$

The number of missing terms in $\mu(u)$ is related in a simple way to the number of missing terms in the product of this function by $(1+u)$. We distinguish two cases. If the first n coefficients of $\mu(u)$ are zero, then they are likewise zero in the product. If, on the other hand, at least n consecutive coefficients in $\mu(u)$ are zero, and if furthermore this gap is preceded by nonzero terms, then in the product at least $n-1$ consecutive coefficients are zero and this gap must also be preceded by nonzero terms. We show that each of these cases leads to a contradiction. We may suppose that not all of $b_0, b_1, \cdots, b_{n-2}$ are zero, for otherwise the problem reduces to the case of n equal to unity. Thus we see from (5) that $(1+u)\mu(u)$ is the sum of a constant and a function which is subject to our induction hypotheses. This function can have at most $n-2$ consecutive missing terms. We have seen that if $\mu(u)$ is to lack n consecutive terms, $(1+u)\mu(u)$ must lack at least $n-1$ such terms. We have also seen that in (5) the first summand can have at most $n-2$ missing terms. We conclude that it is the addition of b_{n-1} to this summand which contributes another zero coefficient. But this can only be the case if it is the first $n-1$ terms of

$(1+u)\mu(u)$ which are zero. In this case our argument shows that if $\mu(u)$ lacked n consecutive terms, $(1+u)\mu(u)$ would also have n consecutive missing terms. This discrepancy proves the lemma.

We are now in a position to prove our partial converse to the result established above. Let θ be a function of x, y, and the y_j, such that the relations $E_t\theta=0$, $t=1, 2, \cdots, n$, are identities in x and y. If the partial derivatives of θ with respect to all the y_j vanish, then θ is a function of x alone, and is certainly an exact nth derivative. We assume henceforth that not all these partial derivatives of θ vanish.

Let m be the largest integer for which $\partial\theta/\partial y_m \neq 0$. Assuming that m is not less than n, we are going to show that there exists a function ψ such that $\theta - D^n\psi$ contains effectively no derivative y_j of order greater than $n-1$. For this purpose we prove first that θ is linear in y_m, and that in θ the coefficient $\partial\theta/\partial y_m$ of y_m is free of $y_{m-1}, y_{m-2}, \cdots, y_{m-n+1}$.

It follows from the theory of determinants that there exist constants $b_1, b_2, \cdots, b_n$, not all zero, such that $b_1E_1+b_2E_2+\cdots+b_nE_n$ is free of $D^i\partial/\partial y_i$ where i has values from $m-n+1$ to $m-1$ inclusive. By virtue of our lemma, this linear combination of the E_i must contain effectively the expression $D^m\partial/\partial y_m$. We suppose the b_i selected in such a way that the coefficient of this expression is unity. Then, because θ does not contain a y_s with s greater than m, we have

$$\sum_{i=1}^{n} b_iE_i\theta = D^m\frac{\partial\theta}{\partial y_m} + cD^{m-n}\frac{\partial\theta}{\partial y_{m-n}} + \cdots + g\frac{\partial\theta}{\partial y}, \tag{6}$$

where $c, \cdots, g$ are constants. The operators E_i are linear, so our assumption that θ is annulled by the E_i implies that both members of (6) are zero. If the term $D^m\partial\theta/\partial y_m$ is zero, it must be that $\partial\theta/\partial y_m$ is a polynomial in x of degree less than m. This certainly agrees with our statement concerning the manner in which y_m is present in θ. Assuming now that this term is not zero, we show that $\partial\theta/\partial y_m$ is free of $y_m, y_{m-1}, \cdots, y_{m-n+1}$. Suppose this is not so. Let y_r be the derivative of greatest order effectively present in $\partial\theta/\partial y_m$. Then y_{m+r} is effectively present in $D^m\partial\theta/\partial y_m$. The other terms of the right member of (6) cannot contain effectively derivatives of y of order greater than $2m-n$, because the partial derivatives of θ are of order at most m and the differentiations increase this order by at most $m-n$. Thus $m+r$ does not exceed $2m-n$, whence r does not exceed $m-n$. This shows that θ differs from $y_m\partial\theta/\partial y_m$ by a function of order less than m, and that $\partial\theta/\partial y_m$ is free of $y_m, y_{m-1}, \cdots, y_{m-n+1}$.

We now introduce an auxiliary function ψ_m described by the relation

$$\psi_m = \int \frac{\partial\theta}{\partial y_m}\, dy_{m-n},$$

where the integration is performed with respect to y_{m-n}, treating $x, y, y_1, \cdots, y_{m-n}$ as independent variables. Because of the use of the indefinite integral in its definition, ψ_m contains an arbitrary additive function of $x, y, y_1, \cdots, y_{m-n-1}$; thus the integration actually leads to many functions, some of which may not possess partial derivatives of all orders. However, the integration also yields functions with partial derivatives of all orders, and it is from these that we suppose a definite ψ_m to be selected. This function also has the property that its partial derivative with respect to y_{m-n} is $\partial\theta/\partial y_m$. Because the order of $\partial\theta/\partial y_m$ is less than $m-n+1$ it follows that $D\psi_m$ differs from $y_{m-n+1}\partial\theta/\partial y_m$ by a function of order less than $m-n+1$. Similarly $D^n\psi_m$ differs from $y_m\partial\theta/\partial y_m$ by a function of order less than m. Then the order of $\theta-D^n\psi_m$ is less than m. In addition, this difference is annulled by all the operators $E_1, \cdots, E_n$. If the order of $\theta-D^n\psi_m$ exceeds $n-1$ we can follow the same procedure with this new function subtracting from it an exact nth derivative and reducing its order still further. After a finite number of steps we obtain a function ψ such that the order of $\theta-D^n\psi$ is less than n.

Let us denote the difference $\theta-D^n\psi$ by τ. It may be that τ is identically zero. In this case θ is an exact nth derivative. Even if τ is not zero, it must still be annulled by $E_1, \cdots, E_n$. Because the order of τ is less than n, the equations $E_t\tau=0$, $t=1, 2, \cdots, n$, constitute a system of n homogeneous linear equations for the n quantities $\partial\tau/\partial y$, $D\partial\tau/\partial y_1, \cdots, D^{n-1}\partial\tau/\partial y_{n-1}$. It follows from the lemma that the determinant of this system is not zero. We conclude that each of the quantities $\partial\tau/\partial y$, $D\partial\tau/\partial y_1, \cdots, D^{n-1}\partial\tau/\partial y_{n-1}$ is zero. Thus τ is a function of the form $\pi_1 y_1+\pi_2 y_2+\cdots+\pi_{n-1}y_{n-1}$ where each π_i is a polynomial in x of degree less than i. We do not overlook the fact that τ could also contain a term free of the y_j, but since such a term, being a function of x alone, would be an exact nth derivative, we may suppose it incorporated into ψ. This completes the proof of the results enunciated at the beginning of the paper.

U. S. NAVY PRE-FLIGHT SCHOOL
IOWA CITY, IOWA