ON EXTENSION OF WRONSKIAN MATRICES

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1. **Introduction.** By an interval J we shall understand a finite interval of the type $a \leqq x \leqq b$. If $u_1(x), u_2(x), \cdots, u_n(x)$ are real functions possessing finite derivatives of the first t orders in an interval J and $0 \leqq s \leqq t$, we call the functional matrix

$$M_s(u_1, \cdots, u_n) \equiv \begin{Vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdots & \cdot \\ u_1^{(s)} & u_2^{(s)} & \cdots & u_n^{(s)} \end{Vmatrix}$$

their Wronskian matrix of order s. The Wronskian $W(u_1, \cdots, u_n)$ is the determinant of the matrix $M_{n-1}(u_1, \cdots, u_n)$.

The principal result we obtain is that if $n \leqq s \leqq t$ and the Wronskian matrix of order s for n arbitrary functions $u_1(x), u_2(x), \cdots, u_n(x)$ of class[1] $C^{(t)}$ in an interval J has constant rank n, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that the extended matrix $M_s(u_1, \cdots, u_{n+1})$ has constant rank $n+1$ in J. We employ a theorem of Curtiss[2] which may be stated in the form:

THEOREM C. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ in an interval J and their Wronskian matrix of order t has rank n throughout J, then the Wronskian $W(u_1, \cdots, u_n)$ has at most isolated zeros.*

From the extension property of Wronskian matrices we obtain a sufficient condition, in terms of the rank of a certain functional matrix, that an arbitrary set of functions having suitable class properties be solutions of an ordinary homogeneous linear differential equation.

2. **Lemmas.** We first prove two lemmas.

LEMMA 1. *If $\delta, c_1, c_2, \cdots, c_n$ are given constants with $\delta > 0$, there exists a function $f(x)$ of class $C^{(n)}$ in the interval $-1 \leqq x \leqq 1$ which satisfies the conditions*: (1) $|f(x)| \leqq \delta$, $-1 \leqq x \leqq 1$; (2) $f^{(i)}(-1) = f^{(i)}(1) = 0$, $i = 0, 1, \cdots, n$; (3) $f(0) = 0$, $f^{(i)}(0) = c_i$, $i = 1, 2, \cdots, n$.

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[1] Each function has continuous derivatives of the first t orders at every point of J.

[2] D. R. Curtiss, *The vanishing of the Wronskian and the problem of linear dependence*, Math. Ann. vol. 65 (1908) Theorem 4.

PROOF. Let $p(x)$ be the polynomial

$$c_1 x + (c_2/2!)x^2 + \cdots + (c_n/n!)x^n.$$

Then $p(0)=0$, $p^{(i)}(0)=c_i$ and for some β with $0<\beta<1$, $|p(x)|\leqq\delta$, $-\beta\leqq x\leqq\beta$. Take α such that $0<\alpha<\beta$. Let

$$\theta = (-1)^n(x+\alpha)^{n+1}/(x+\beta)^2, \qquad \phi = -(x-\alpha)^{n+1}/(x-\beta)^2,$$

and let

$$g_1(x) \equiv \begin{cases} 0, & x \leqq -\beta, \\ e^\theta, & -\beta < x < -\alpha, \\ 1, & -\alpha \leqq x \leqq 0, \end{cases} \qquad g_2(x) \equiv \begin{cases} 1, & 0 \leqq x \leqq \alpha, \\ e^\phi, & \alpha < x < \beta, \\ 0, & \beta \leqq x. \end{cases}$$

The g's are of class $C^{(n)}$ in their respective intervals of definition and $g_1^{(i)}(0)=g_2^{(i)}(0)$ for $i=0, 1, \cdots, n$. Then the function

$$f(x) \equiv \begin{cases} p(x)g_1(x), & -1 \leqq x \leqq 0, \\ p(x)g_2(x), & 0 \leqq x \leqq 1, \end{cases}$$

is of class $C^{(n)}$ in the interval $-1\leqq x\leqq 1$ and satisfies the prescribed conditions.

LEMMA 2. *Given an interval J and constants a_i and b_i, $i=0, 1, \cdots, n$ with $a_0b_0>0$, there exists a function $f(x)$ which satisfies the conditions: (1) $f(x)$ is nonzero and of class $C^{(n)}$ in J; (2) $f^{(i)}(a)=a_i$, $f^{(i)}(b)=b_i$, $i=0, 1, \cdots, n$.*

PROOF. Let $\delta=1/3 \min(|a_0|, |b_0|, b-a)$. Take $f_k(x)$, $k=1, 2$, of class $C^{(n)}$ in the interval $-1\leqq x\leqq 1$ such that: $|f_k(x)|\leqq\delta$, $-1\leqq x\leqq 1$; $f_k^{(i)}(-1)=f_k^{(i)}(1)=f_k(0)=0$, $i=0, 1, \cdots, n$; $f_1^{(i)}(0)=a_i$, $f_2^{(i)}(0)=b_i$, $i=1, 2, \cdots, n$. Let

$$u(x) \equiv f_1((x-a)/\delta) + a_0, \qquad a \leqq x \leqq a+\delta,$$
$$v(x) \equiv f_2((x-b)/\delta) + b_0, \qquad b-\delta \leqq x \leqq b.$$

Then $u(a)=u(a+\delta)=a_0$; $|u(x)-u(a)|\leqq\delta$, $a\leqq x\leqq a+\delta$; $u^{(i)}(a)=a_i$, $u^{(i)}(a+\delta)=0$, $i=1, 2, \cdots, n$; and $v(b)=v(b-\delta)=b_0$; $|v(x)-v(b)| \leqq\delta$, $b-\delta\leqq x\leqq b$; $v^{(i)}(b)=b_i$, $v^{(i)}(b-\delta)=0$, $i=1, 2, \cdots, n$. Moreover, both $u(x)$ and $v(x)$ are nonzero in their respective mutually exclusive intervals of definition by our choice of δ. Let $w(x)\equiv(a_0-b_0)e^\theta+b_0$, $a+\delta\leqq x\leqq b-\delta$, where $\theta=-(x-a-\delta)^{n+1}/(x-b+\delta)^2$. Then $w(x)$ is nonzero throughout its interval of definition and

$$w(a+\delta)=a_0,\ w(b-\delta)=b_0,\ w^{(i)}(a+\delta)=w^{(i)}(b-\delta)=0, \qquad i=1, 2, \cdots, n.$$

Therefore the function

$$f(x) \equiv \begin{cases} u(x), & a \leqq x \leqq a+\delta, \\ w(x), & a+\delta \leqq x \leqq b-\delta, \\ v(x), & b-\delta \leqq x \leqq b, \end{cases}$$

satisfies the conditions of the lemma.

3. **Extension of Wronskian matrices.** Let $u(x)$ be a function of class $C^{(t)}$, $t \geqq 1$, in an interval J with $M_1(u)$ of rank 1 at every point of J. If $v(x)$ of class $C^{(t)}$ exists such that $M_1(u, v)$ has rank 2 throughout J, the Wronskian $W(u, v)$ is nonzero in J. Thus in this case there exists a function $K(x)$ of class $C^{(t-1)}$ and nonzero in J such that $v(x)$ is a solution of the differential equation

$$u(x)y' - u'(x)y = K(x). \tag{1}$$

This leads us to ask: does (1) have a solution of class $C^{(t)}$ in J for arbitrary $K(x)$ of class $C^{(t-1)}$ which is nonzero in J? We give a counter-example.

Example. Let J be the interval $0 \leqq x \leqq 2$ and take $u(x) = 1-x$, $K(x) = 1 + |1-x|^{1/2}$. Then the differential equation (1) becomes

$$(1-x)y' + y = 1 + |1-x|^{1/2}, \tag{2}$$

for which the general solution in the interval $0 \leqq x < 1$ is

$$y(x) = u(x)\int_0^x (K(z)/u^2(z))dz + cu(x)$$

where c is constant. Thus

$$y'(x) = (1/u(x))[K(x) + u'(x)y(x)] \equiv 3 - c - (1-x)^{-1/2}$$

and $\lim_{x \to 1-0} y'(x)$ does not exist, so that the differential equation (2) has no solution of class C' defined in the interval $0 \leqq x \leqq 2$.

We now consider the special case of extending a Wronskian matrix of n columns and $n+1$ rows which has rank n throughout an interval J.

THEOREM 1. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ $(t \geqq n)$ in an interval J and their Wronskian matrix of order n has rank n at every point of J, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that $M_n(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout J.*

PROOF. Since $M_n(u_1, \cdots, u_n)$ has rank n throughout J, it follows from Theorem C that $W(u_1, \cdots, u_n)$ has at most a finite number of zeros in J. Denote by

$$(3) \qquad f_r(x), \qquad r = 0, 1, \cdots, n,$$

the cofactor of $y^{(r)}$ in the determinant $W(u_1, \cdots, u_n, y)$ in which y is unknown. If we prove the existence of a function $K(x)$ which is nonzero and of class $C^{(t-n)}$ in J for which the differential equation

$$(4) \quad f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = K(x)$$

has a solution defined in J, we shall have proved our theorem since the coefficients in (4) are of class $C^{(t-n)}$. If $f_n(x)$ does not vanish in J, standard existence theorems apply for arbitrary $K(x)$.

Let $f_n(x)$ vanish in J. Since $f_n(x) \equiv W(u_1, \cdots, u_n)$, its zeros are finite in number. Assume that the zeros are at the points $c_1, c_2, \cdots, c_m$ with $c_1 < c_2 < \cdots < c_m$. If the c's are not all interior points of J, let

$$U_i(x) \equiv \begin{cases} \sum_{j=0}^{t} \frac{1}{j!} u_i^{(j)}(a)(x-a)^j, & x < a, \\ u_i(x), & a \leqq x \leqq b, \qquad i = 1, 2, \cdots, n. \\ \sum_{j=0}^{t} \frac{1}{j!} u_i^{(j)}(b)(x-b)^j, & x > b, \end{cases}$$

Then $W(U_1, \cdots, U_n)$ is identical with $W(u_1, \cdots, u_n)$ in J, and is a polynomial for x not in J. Therefore $W(U_1, \cdots, U_n)$ has isolated zeros and for some $a^* < a$ and $b^* > b$, the c's are its only zeros in the interval $a^* \leqq x \leqq b^*$. Consequently we assume without loss of generality that $c_1 > a$ and $c_m < b$.

Since $M_n(u_1, \cdots, u_n)$ has rank n throughout J, for each i, $i = 1, 2, \cdots, m$, at least one function in (3) does not vanish at c_i. Denote by $f_{k_i}(x)$ a function from (3) such that $f_{k_i}(c_i) \neq 0$, $i = 1, 2, \cdots, m$. Define

$$q_i(x) \equiv \pm (1/k_i!)(x - c_i)^{k_i}, \qquad i = 1, 2, \cdots, m,$$

where the ambiguous sign is so chosen that each Wronskian $W(u_1, \cdots, u_n, q_i)$, which has the value $\pm f_{k_i}(c_i)$ at c_i, has the same sign at c_i. Then, for some $\epsilon > 0$ with $a < c_1 - \epsilon < c_1 + \epsilon < c_2 - \epsilon < \cdots < c_m + \epsilon < b$,

$$Q_i(x) \equiv W(u_1, \cdots, u_n, q_i) \neq 0, c_i - \epsilon \leqq x \leqq c_i + \epsilon, i = 1, 2, \cdots, m,$$

and $q_i(x)$ is a solution of the differential equation

$$W(u_1, \cdots, u_n, y) = Q_i(x), \qquad i = 1, 2, \cdots, m.$$

Denote the nterval

$$(5) \qquad c_{k-1} + \epsilon \leqq x \leqq c_k - \epsilon, \quad k = 1, 2, \cdots, m+1,$$

where $c_0+\epsilon=a$ and $c_{m+1}-\epsilon=b$, by J_k. Let $P_k(x)$ be any function (Lemma 2) which is nonzero and of class $C^{(t-n)}$ in J_k and satisfies the conditions

$$(6) \qquad \begin{aligned} P_k^{(j)}(c_{k-1}+\epsilon+0) &= Q_{k-1}^{(j)}(c_{k-1}+\epsilon-0), \\ P_k^{(j)}(c_k-\epsilon-0) &= Q_k^{(j)}(c_k-\epsilon+0), \end{aligned}$$

for $j=0, 1, \cdots, t-n$, where

$$\begin{aligned} Q_0^{(j)}(c_0+\epsilon-0) &= Q_1^{(j)}(c_1-\epsilon+0), \\ Q_{m+1}^{(j)}(c_{m+1}-\epsilon+0) &= Q_m^{(j)}(c_m+\epsilon-0). \end{aligned}$$

Then each function

$$K_k(x) \equiv \begin{cases} P_k(x), & c_{k-1}+\epsilon \leqq x \leqq c_k-\epsilon, \\ Q_k(x), & c_k-\epsilon \leqq x \leqq c_k+\epsilon, \end{cases} \qquad k = 1, 2, \cdots, m+1,$$

where $c_{m+1}+\epsilon=b$, is nonzero and of class $C^{(t-n)}$ in its interval of definition by the second set of conditions in (6). Thus the function $K(x)$ defined in J by

$$(7) \quad K(x) \equiv K_k(x), \quad c_{k-1}+\epsilon \leqq x \leqq c_k+\epsilon, \quad k = 1, 2, \cdots, m+1,$$

is nonzero and of class $C^{(t-n)}$ in J by the first set of conditions in (6).

Since $f_n(x)\neq 0$ in J_1, the differential equation $W(u_1, \cdots, u_n, y) = P_1(x)$ has a solution $p_1(x)$ which satisfies the initial conditions

$$(8) \qquad p_1^{(j)}(c_1-\epsilon-0) = q_1^{(j)}(c_1-\epsilon+0), \qquad j = 0, 1, \cdots, n-1.$$

It follows from the second set of conditions in (6) that (8) also holds for $j=n, n+1, \cdots, t$. The function

$$y_1(x) \equiv \begin{cases} p_1(x), & a \leqq x \leqq c_1-\epsilon, \\ q_1(x), & c_1-\epsilon \leqq x \leqq c_1+\epsilon, \end{cases}$$

is therefore a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_1(x)$.

In J_2 the differential equation

$$(9) \qquad W(u_1, \cdots, u_n, y) = P_2(x)$$

has a solution $p_2(x)$ with $p_2^{(j)}(c_1+\epsilon+0)=y_1^{(j)}(c_1+\epsilon-0), j=0, 1, \cdots, t,$

and also a solution $r(x)$ with $r^{(j)}(c_2-\epsilon-0)=q_2^{(j)}(c_2-\epsilon+0)$, $j=0, 1, \cdots, t$. Now the difference of any two solutions of (9) is a solution of the corresponding homogeneous equation for which the u's are linearly independent solutions. Thus, for some set of constants $a_1, a_2, \cdots, a_n$,

$$p_2(x) \equiv r(x) + a_1u_1(x) + \cdots + a_nu_n(x), \quad c_1 + \epsilon \leqq x \leqq c_2 - \epsilon.$$

Then the function

$$y_2(x) \equiv \begin{cases} p_2(x), & c_1 + \epsilon \leqq x \leqq c_2 - \epsilon, \\ q_2(x) + a_1u_1(x) + \cdots + a_nu_n(x), & c_2 - \epsilon \leqq x \leqq c_2 + \epsilon, \end{cases}$$

is of class $C^{(t)}$ in its interval of definition with

$$y_2^{(j)}(c_1 + \epsilon + 0) = y_1^{(j)}(c_1 + \epsilon - 0), \qquad j = 0, 1, \cdots, t,$$

and is a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_2(x)$.

We continue in the above manner, obtaining $y_k(x)$, $c_{k-1}+\epsilon \leqq x \leqq c_k + \epsilon$ $(k=2, 3, \cdots, m+1)$ of class $C^{(t)}$ in its interval of definition with

$$y_k^{(j)}(c_{k-1} + \epsilon + 0) = y_{k-1}^{(j)}(c_{k-1} + \epsilon - 0), \qquad j = 0, 1, \cdots, t,$$

which is a solution of the differential equation $W(u_1, \cdots, u_n, y) = K_k(x)$. Then the function $u_{n+1}(x)$ defined in J by

$$u_{n+1}(x) \equiv y_k(x), \quad c_{k-1} + \epsilon \leqq x \leqq c_k + \epsilon, \quad k = 1, 2, \cdots, m + 1,$$

is of class $C^{(t)}$ in J and is a solution of (4) with $K(x)$ defined by (7).

In the following theorem we consider the general case of extending a Wronskian matrix.

THEOREM 2. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ $(t \geqq n)$ in an interval J and their Wronskian matrix of order s $(n \leqq s \leqq t)$ has rank n at every point of J, there exists a function $u_{n+1}(x)$ of class $C^{(t)}$ such that $M_s(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout J.*

PROOF. The case $s=n$ is treated in Theorem 1. Assume now that $M_n(u_1, \cdots, u_n)$ does not have rank n throughout J. Since $M_s(u_1, \cdots, u_n)$ has rank n at every point, it follows from Theorem C that $W(u_1, \cdots, u_n)$ has isolated zeros. Define $f_r(x)$, $r=0, 1, \cdots, n$, as in (3). Since the f's are the n-rowed minor determinants of $M_n(u_1, \cdots, u_n)$ they have at least one zero in common, but only a finite number for $f_n(x) \equiv W(u_1, \cdots, u_n)$. Assume that the common zeros of the f's are at the points $c_1, c_2, \cdots, c_m$ with $c_1 < c_2 < \cdots < c_m$. There is no loss of generality to assume that $c_1 > a$ and $c_m < b$.

For each i, $i=1, 2, \cdots, m$, some n-rowed determinant of $M_s(u_1, \cdots, u_n)$, which we shall denote by Δ_i, does not vanish at c_i. If $u(x)$ is a function of class $C^{(t)}$ in J and row r of $M_s(u_1, \cdots, u_n)$ is not a row of Δ_i, denote by $\Delta_{i,r}$ the determinant of $M_s(u_1, \cdots, u_n, u)$ which consists of row r and the rows represented in Δ_i.

Let row r_i be the first row of $M_s(u_1, \cdots, u_n)$ which is not a row of Δ_i. Define

$$q_i(x) \equiv (1/(r_i - 1)!)(x - c_i)^{r_i-1}, \qquad i = 1, 2, \cdots, m.$$

Then at c_i the determinant Δ_{i,r_i} of $M_s(u_1, \cdots, u_n, q_i)$ has the value $(-1)^{n+1+r_i}\Delta_i(c_i)$, hence for some $\delta>0$ with $a<c_1-\delta<c_1+\delta<c_2-\delta < \cdots <c_m+\delta<b$,

$$(10) \qquad \Delta_{i,r_i} \neq 0, \; c_i - \delta \leqq x \leqq c_i + \delta, \qquad i = 1, 2, \cdots, m.$$

Since $M_s(u_1, \cdots, u_n, q_i)$ has rank $n+1$ throughout the interval $c_i-\delta\leqq x\leqq c_i+\delta$ by (10), $W(u_1, \cdots, u_n, q_i)$ has isolated zeros in this interval. Thus, for some $\epsilon>0$ with $\epsilon\leqq\delta$,

$$(11) \quad Q_i(x) \equiv W(u_1, \cdots, u_n, q_i) \neq 0 \text{ at } x = c_i \pm \epsilon, \; i = 1, 2, \cdots, m.$$

We shall assume that $Q_{i+1}(c_{i+1}-\epsilon)$, $i=1, 2, \cdots, m-1$, is positive or negative according as $Q_i(c_i+\epsilon)$ is positive or negative, since $q_{i+1}(x)$ could be replaced by $-q_{i+1}(x)$ and the expressions corresponding to (10) and (11) would hold.

Define the interval J_k as in (5). Since $f_0(x), f_1(x), \cdots, f_n(x)$ have no common zero in J_k, $M_n(u_1, \cdots, u_n)$ has rank n at every point of J_k. Thus, by the method employed in the proof of Theorem 1, we can define a function $P_k(x)$, $k=1, 2, \cdots, m+1$, which is nonzero and of class $C^{(t-n)}$ in J_k, which satisfies the conditions

$$(12) \qquad \begin{aligned} P_k^{(j)}(c_{k-1} + \epsilon + 0) &= Q_{k-1}^{(j)}(c_{k-1} + \epsilon - 0), \\ P_k^{(j)}(c_k - \epsilon - 0) &= Q_k^{(j)}(c_k - \epsilon + 0) \end{aligned}$$

for $j=0, 1, \cdots, t-n$, where

$$\begin{aligned} Q_0^{(j)}(c_0 + \epsilon - 0) &= Q_1^{(j)}(c_1 - \epsilon + 0), \\ Q_{m+1}^{(j)}(c_{m+1} - \epsilon + 0) &= Q_m^{(j)}(c_m + \epsilon - 0), \end{aligned}$$

and for which the differential equation $W(u_1, \cdots, u_n, y)=P_k(x)$, $k=1, 2, \cdots, m+1$, has a solution of class $C^{(t)}$ defined in J_k.

In J_1 the differential equation

$$(13) \qquad W(u_1, \cdots, u_n, y) = P_1(x)$$

has a solution $p_1(x)$ with $p_1^{(j)}(c_1-\epsilon-0)=q_1^{(j)}(c_1-\epsilon+0)$, $j=0, 1, \cdots, t$, in view of the second set of conditions in (12). Then the function

$$y_1(x) \equiv \begin{cases} p_1(x), & a \leqq x \leqq c_1 - \epsilon, \\ q_1(x), & c_1 - \epsilon \leqq x \leqq c_1 + \epsilon, \end{cases}$$

is of class $C^{(t)}$, and $M_s(u_1, \cdots, u_n, y_1)$ has rank $n+1$ at every point of the interval $a \leqq x \leqq c_1+\epsilon$ since $p_1(x)$ is a solution of (13) and the determinant Δ_{1,r_1} does not vanish in the interval $c_1-\epsilon \leqq x \leqq c_1+\epsilon$ by (10).

In J_2 the differential equation

$$W(u_1, \cdots, u_n, y) = P_2(x) \tag{14}$$

has a solution $p_2(x)$ with $p_2^{(j)}(c_1+\epsilon+0)=y_1^{(j)}(c_1+\epsilon-0)$, $j=0, 1, \cdots, t$, and also a solution $r(x)$ with $r^{(j)}(c_2-\epsilon-0)=q_2^{(j)}(c_2-\epsilon+0)$, $j=0, 1, \cdots, t$. Since $u_1(x), u_2(x), \cdots, u_n(x)$ are linearly independent solutions of the homogeneous equation corresponding to (14), for some set of constants $a_1, a_2, \cdots, a_n$,

$$p_2(x) \equiv r(x) + a_1u_1(x) + \cdots + a_nu_n(x), \quad c_1 + \epsilon \leqq x \leqq c_2 - \epsilon.$$

Then the function

$$y_2(x) \equiv \begin{cases} p_2(x), & c_1 + \epsilon \leqq x \leqq c_2 - \epsilon, \\ q_2(x) + a_1u_1(x) + \cdots + a_nu_n(x), & c_2 - \epsilon \leqq x \leqq c_2 + \epsilon, \end{cases}$$

is of class $C^{(t)}$ in its interval of definition with

$$y_2^{(j)}(c_1 + \epsilon + 0) = y_1^{(j)}(c_1 + \epsilon - 0), \quad j = 0, 1, \cdots, t,$$

and $M_s(u_1, \cdots, u_n, y_2)$ has rank $n+1$ throughout the interval $c_1+\epsilon \leqq x \leqq c_2+\epsilon$, since $p_2(x)$ is a solution of (14) and the determinant Δ_{2,r_2} does not vanish in the interval $c_2-\epsilon \leqq x \leqq c_2+\epsilon$ by (10).

We continue in the above manner, obtaining $y_k(x)$, $c_{k-1}+\epsilon \leqq x \leqq c_k+\epsilon$ ($k=2, 3, \cdots, m+1$, where $c_{m+1}+\epsilon=b$) of class $C^{(t)}$ in its interval of definition with

$$y_k^{(j)}(c_{k-1} + \epsilon + 0) = y_{k-1}^{(j)}(c_{k-1} + \epsilon - 0), \quad j = 0, 1, \cdots, t,$$

and with each matrix $M_s(u_1, \cdots, u_n, y_k)$ of rank $n+1$ throughout the interval $c_{k-1}+\epsilon \leqq x \leqq c_k+\epsilon$. Then the function $u_{n+1}(x)$ defined in J by

$$u_{n+1}(x) \equiv y_k(x), \quad c_{k-1} + \epsilon \leqq x \leqq c_k + \epsilon, \quad k = 1, 2, \cdots, m + 1,$$

is of class $C^{(t)}$ in J and the Wronskian matrix $M_s(u_1, \cdots, u_{n+1})$ has rank $n+1$ throughout J.

4. Application to the theory of differential equations. The extension property of Wronskian matrices leads us to the following sufficient condition that a set of n given functions be solutions of an ordinary homogeneous linear differential equation.

THEOREM 3. *If $u_1(x), u_2(x), \cdots, u_n(x)$ are functions of class $C^{(t)}$ $(t \geqq n)$ in an interval J and their Wronskian matrix of order s $(s<t)$ has rank n at every point of J, then the u's are linearly independent solutions of a homogeneous linear differential equation of order $s+1$ of the type*

$$y^{(s+1)} + f_s(x)y^{(s)} + \cdots + f_1(x)y' + f_0(x)y = 0 \tag{15}$$

in which the $f_i(x)$ are functions of class $C^{(t-s-1)}$ in J.

PROOF. Let $m=s-n+1$. The case $m=0$ is well known and $m<0$ is impossible. Assume now that $m>0$. Then $n \leqq s<t$ and, by m successive applications of Theorem 2, there exist m functions, $u_{n+1}(x), u_{n+2}(x), \cdots, u_{n+m}(x)$, of class $C^{(t)}$ in J such that the Wronskian matrices

$$M_s(u_1, \cdots, u_{n+1}),\ M_s(u_1, \cdots, u_{n+2}),\ \cdots,\ M_s(u_1, \cdots, u_{n+m}) \tag{16}$$

have the respective ranks $n+1, n+2, \cdots, n+m$ at every point of J. Since $n+m=s+1$, the last matrix in (16) is the Wronskian matrix of order s for $s+1$ functions of class $C^{(t)}$ $(t>s)$ with $W(u_1, \cdots, u_{s+1}) \neq 0$ throughout J. Therefore $u_1(x), u_2(x), \cdots, u_{s+1}(x)$ constitute a fundamental system of solutions of a homogeneous equation of type (15) with the $f_i(x)$ of class $C^{(t-s-1)}$ in J. Hence the n given functions have the asserted property.

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