ON THE CROSSING OF EXTREMALS AT FOCAL POINTS

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Morse and Littauer[1] have proved the following theorem for an analytic Finsler space, where g is an extremal transversal to the (analytic) hypersurface Σ.

THEOREM. *A necessary and sufficient condition that p on g be a focal point of Σ is that the family of extremals cut transversaly by Σ near g shall fail to cover the neighborhood of p simply.*

The purpose of the present paper is to prove this theorem on the weaker hypothesis that the Finsler space and Σ are of class C'''.

As pointed out in M. L. the sufficiency of the condition is trivial, and in proving the necessity there is no loss in generality if we assume p to be a first focal point. It is further clear from M. L. that the theorem is a consequence of the following lemma.

LEMMA I. *If p is a first focal point on g contained in a (simply covering) field R of extremals transversal to Σ, then there exists a first focal point q covered by R and a subfield S of R covering q and such that the Hilbert integral is independent of path for paths confined to S.*

Before proceeding to the proof of Lemma I we will establish a secondary lemma.

LEMMA II. *Let T be a transformation of class C' mapping a closed coordinate neighborhood A into a closed Riemannian manifold B, then almost all points of B (in the measure theoretic sense) have finite counter images.*

PROOF. Call the set of points $K \subset A$ at which the Jacobian of T vanishes critical points, then I assert that if the counter image $T^{-1}b$, $b \in B$, is infinite, it contains a critical point. In fact if b^i are the coordinates of such a point b, there is a convergent sequence of points of $T^{-1}b$ with coordinates a_σ^i approaching a point a_0 from a definite direction, as is expressed by the following set of equations.

$$
\begin{gathered}
\overset{i}{a}_\sigma \to \overset{i}{a}_0,\ \overset{i}{\xi}_\sigma = (\overset{i}{a}_\sigma - \overset{i}{a}_0)/(\Sigma_i(\overset{i}{a}_\sigma - \overset{i}{a}_0)^2)^{1/2} \to \overset{i}{\xi}_0, \\
T^i(\overset{j}{a}_\sigma) = \overset{i}{b} = T^i(\overset{j}{a}_0).
\end{gathered} \tag{1}
$$

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[1] Marston Morse and S. B. Littauer, *A characterization of fields in the calculus of variations*, Proc. Nat. Acad. Sci. U.S.A. vol. 18 (1932) pp. 724–730. This paper will hereafter be designated by M. L.

By the theorem of the mean we have

$$(2)\quad T^{i}(a^{j}_{\sigma}) = T^{i}(a^{j}_{0}) + \Sigma_{k}\frac{\partial T^{i}(a^{j}_{0} + \theta^{i}_{\sigma}(a^{j} - a^{j}_{\sigma}))}{\partial a^{k}}(a^{j}_{\sigma} - a^{j}_{0}),\ 0 \leqq \theta^{i}_{\sigma} \leqq 1,$$

finally

$$(3)\qquad \Sigma_{k}\frac{\partial T^{i}(a^{j}_{0})}{\partial a^{k}}\xi^{k}_{0} = 0$$

so $a_0 \in K$.

It remains to be shown that TK is of measure zero. But this follows quite easily from the measure theoretic significance of the Jacobian of T.

PROOF OF LEMMA I. Let $s=(s_1, \cdots, s_{n-1})$ denote coordinates of points of Σ relative to some admissible coordinate system of Σ covering the intersection of g and Σ at s_0, and along any geodesic transversal to Σ in this neighborhood let r denote (Finsler) length from Σ (considered positive in the direction of p). Now according to the hypothesis of the lemma if r_0 is the arc length along g from Σ to p, there is a δ such that for $|s^i - s^i_0| \leqq \delta$ and $-\delta \leqq r \leqq r_0 + \delta$ the numbers (r, s) constitute a closed coordinate neighborhood U of class C^0.

Of course the coordinate neighborhood U is not admissible. In fact if X denotes an admissible coordinate neighborhood of p and P the natural transformation from U to X, then the fact that p is a focal point is expressed by the vanishing of the Jacobian $\Delta(r,s)$ of P at r_0, s_0.

We are assuming that p is a first focal point. According to a technique of Morse[2] the locus of first focal points near p is a hypersurface determined by an equation of the form $r=f(s)$. Now I assert that there is a point q with coordinates $r_1=f(s_1)$, such that, in a neighborhood of q, f is of class C', and all focal points in that neighborhood lie on the locus $r=f(s)$. To find such a point choose s_1 in such a way as to maximize the rank of the determinant $\Delta(f(s), s)$ in the neighborhood of s_0. Call this rank k. By slightly sharpening a result of Morse[3] we can find a $(k+1)$ minor $\Delta^*(r, s)$ of $\Delta(r, s)$ such that $\partial\Delta^*(f(s), s)/\partial r \neq 0$. From the assumption that the rank of $\Delta(f(s), s)$ is at most k in the neighborhood of s_1 it follows that $\Delta^*(f(s), s) \equiv 0$ in that neighborhood, but now the implicit function theorem applies to show that f is of class C' near s_1. Finally we conclude, following Morse,[4] that all

[2] Cf. Marston Morse, *The calculus of variations in the large*, Amer. Math. Soc. Colloquium Publications vol. 18 (1934) p. 235 Lemma 13.1.

[3] Marston Morse, *The order of vanishing of the determinant of a conjugate base*, Proc. Nat. Acad. Sci. U.S.A. vol. 17 (1931) pp. 319–320.

[4] See *Calculus of variations in the large*, loc. cit.

focal points in the neighborhood of q lie on the locus $r=f(s)$.

We are now in a position to construct the following set-up. Let Y be an admissible coordinate neighborhood containing q and the extremal of R joining q to Σ. Let S be a closed subfield of R, contained in Y, covering q and such that the only focal points of Σ within S satisfy $r=f(s)$. Call this intersection of the focal point locus with S π. Let V denote the sub-neighborhood of U covered by S.

Now we must consider the Hilbert integral in the region covered by S. This is a certain line integral I to be evaluated with respect to the coordinate system Y. The corresponding integral I^* with respect to V turns out according to M. L. to be independent of path, and we would like to make the same conclusion about I. The difficulty lies in the fact that though the natural transformation Q from V to Y is of class C', its inverse is differentiable only on the complement of $Q\pi$. We can get around this by approximating arbitrary curves of class C' in $Y \cap S$ in the derivative by rectilinear polygons (with respect to Y) which intersect $Q\pi$ in only a finite number of points. This is easily achieved by means of Lemma II, from which we can conclude that if a point is not on $Q\pi$, then almost all rays (with respect to Y) issuing from it intersect $Q\pi$ in only a finite number of points. Q^{-1} carries such polygons into curves differentiable at all but a finite number of points. I^* can be extended to these curves by means of Riemann-Cauchy integration. The proof of the lemma is then easily completed.

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