# A FAMILY OF FUNCTIONS AND ITS THEORY OF CONTACT[1]

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**Introduction.** If $p_1, \cdots, p_n$ are fixed positive integers and $a_1, \cdots, a_n$ arbitrary constants, it is possible so to choose the $a_i$ as to make the function

$$(1) \qquad y(x) = \prod_{i=1}^{n} (x - a_i)^{p_i}$$

and its first $p_1 + \cdots + p_n - 1$ derivatives equal to zero for any single value $x_0$ of $x$. This is accomplished by taking each $a_i$ equal to $x_0$. One might say, on this basis, that *the family of polynomials* (1) *has contact of order* $p_1 + \cdots + p_n - 1$, *for every value of* $x$, *with* $y = 0$.

A more interesting situation is met when we allow the $p_i$ to be any fixed positive numbers, not necessarily integral. In that case $y(x)$ may be a function of many branches, with the quotient of any two branches equal to a constant of modulus unity. For our purposes it suffices to consider the value zero of $x$. If no $a_i$ is zero, each branch of $y(x)$ will be analytic at $x = 0$, with an expansion

$$c_0 + c_1 x + \cdots + c_s x^s + \cdots$$

where the $c_j$ depend on the $a_i$. The question which we examine is: *What is the greatest value of* $s$ *such that, by suitably varying the* $a_i$, *the coefficients* $c_0, \cdots, c_s$ *can be made to approach zero simultaneously*? Such a greatest value of $s$ exists, and will be called, below, *the order of contact of the family* (1) *with* $y = 0$. Denoting the greatest value of $s$ by $r$, we shall prove that

$$(2) \qquad r \leqq q + n - 1$$

where $q$ is the greatest integer less than $p_1 + \cdots + p_n$. When no proper subset of the $p_i$ has an integral sum, the equality sign holds in (2). For $n = 2$, (2) can be an inequality only when $p_1$ and $p_2$ are both integers. For $n \geqq 3$, (2) will certainly be an inequality if some integral power of $y(x)$ is a polynomial of degree not exceeding $q + n - 1$; thus the order of contact of the family

Received by the editors April 9, 1942.

[1] The problem of this note was suggested by the considerations of our paper *On the singular solutions of algebraic differential equations*, Ann. of Math. (2) vol. 37 (1936) p. 552. See also, W. C. Strodt, Trans. Amer. Math. Soc. vol. 45 (1939) p. 276.

$$y(x) = (x - a_1)^{1/2}(x - a_2)^{1/2}(x - a_3)^{1/2}$$

is two rather than three. Whether this describes all exceptional cases for $n \geqq 3$ is not decided here.

1. **The family of functions.** In what follows, the $p_i$ in (1) will be any fixed positive numbers. A few words are necessary to make clear the meaning of the second member of (1) for given $a_i$. If the $a_i$ are distinct from one another, we may take any simply connected area containing no $a_i$ and form the product, in the area, of any selection of branches of the $n$ functions $(x-a_i)^{p_i}$. The various products obtainable in this way are continuations of one another and are all branches of a single analytic function, which we consider the second member of (1) to represent. If two or more $a_i$ coincide, two distinct products, as just described, need not be branches of the same analytic function. There may thus be more than one, possibly even a countable infinitude of interpretations of the second member of (1); every such analytic function will be accepted into the $n$-parameter family of functions (1).

Given any function $y$, as in (1), its values, for any $x$ which is not an $a_i$, are equal in modulus; the same is true for every derivative of $y$.

2. **Order of contact.** Let $\mathcal{J}$ be a family of analytic functions and $f(x)$ a function[2] analytic at a point $x_0$. There may exist non-negative integers $s$ which have the property that, for every $\epsilon > 0$, a $g(x)$ exists in $\mathcal{J}$, with a branch analytic at $x_0$, such that, for this branch of $g(x)$, $g(x)-f(x)$ and its first $s$ derivatives are less than $\epsilon$ in modulus at $x_0$. If such integers $s$ exist, and if the set of them is bounded, we shall represent the greatest of them by $r$ and shall say that $\mathcal{J}$ has *contact of order r* with $f(x)$ at $x_0$. If the $s$ are unbounded, we shall say that $\mathcal{J}$ has contact of infinite order with $f(x)$ at $x_0$.

3. **The bound.** We examine now the functions (1). It is apparent that this family has contact of some order with $y=0$ at every point. Indeed, because the family is invariant under the addition of any constant to $x$, the contact with $y=0$ is the same for all values of $x$.

Let $q$ be the greatest integer less than $p_1 + \cdots + p_n$. The order of contact of the family (1) with $y=0$ is not less than $q$. This is seen by taking all $a_i$ equal to zero. We prove the theorem:

THEOREM. *The order of contact which the family* (1) *has with* $y=0$, *for every* $x$, *does not exceed* $q+n-1$.

[2] Not necessarily in $\mathcal{J}$.

The theorem is readily seen to be true for $n=1$; we employ induction with respect to $n$. We examine the theorem for $n=r>1$, assuming that it has been established for every $n$ less than $r$.

We suppose the theorem false for $n=r$. Then, for $n=r$, and for certain positive numbers $p_1, \cdots, p_r$ which stay fixed during our proof, the family (1) has contact with $y=0$, for $x=0$, of order greater than $q+r-1$. Thus, if we denote the $j$th derivative of $y$ by $y_j$,[3] we can, for every $\epsilon>0$, fix the $a_i$ in (1) at values distinct from 0 so as to have[4]

$$|y_i(0)| < \epsilon, \qquad i = 0, 1, \cdots, q+r. \tag{3}$$

Let us show that, if $\epsilon$ is sufficiently small, each $a_i$, as just fixed, will have a modulus less than unity. Suppose, for instance, that for some very small $\epsilon$, $|a_1| \geqq 1$. Then $y(x)/(x-a_1)^{p_1}$ will be very small, together with its first $q+r$ derivatives, at $x=0$. This, by the case of $n=r-1$, is impossible.

We now put

$$\alpha(x) = (x - a_1) \cdots (x - a_r);$$

$$\beta(x) = \alpha(x)\left[\frac{p_1}{x - a_1} + \cdots + \frac{p_r}{x - a_r}\right]. \tag{4}$$

We have

$$\alpha(x)y_1 - \beta(x)y = 0. \tag{5}$$

The polynomial $\beta$ is of degree $r-1$. Its $(r-1)$st derivative is

$$(r-1)!(p_1 + \cdots + p_r). \tag{6}$$

We differentiate (5) $j-1$ times, where $j \geqq 1$. Indicating derivatives of $\alpha$ and $\beta$ by subscripts, we find that

$$\begin{aligned}\alpha y_j + [(j-1)\alpha_1 - \beta]y_{j-1} + \left[\frac{(j-1)(j-2)}{2!}\alpha_2 - (j-1)\beta_1\right]y_{j-2} \\ + \cdots - \beta_{j-1}y = 0.\end{aligned} \tag{7}$$

For $j \geqq r$, (7) becomes, because of the degrees of $\alpha$ and $\beta$,

$$\alpha y_j + \cdots + (j-1)!\left[\frac{\alpha_r}{r!(j-r-1)!} - \frac{\beta_{r-1}}{(r-1)!(j-r)!}\right]y_{j-r} = 0. \tag{8}$$

[3] $y_0 = y$.

[4] If $y$ is analytic at $x=0$ when certain $h$ of the $a_i$, say $a_1, \cdots, a_h$, are zero while no other $a_i$ vanish, it must be that $p_1 + \cdots + p_h$ is integral. Thus, if $a_1, \cdots, a_h$ are changed to a common value slightly different from zero, $y$ and any specified finite set of its derivatives will undergo only a slight change at $x=0$.

The coefficient of $y_{j-r}$ in (8) is a constant, which, if we have regard to (6) and notice that $\alpha_r = r!$, is seen to be zero if and only if

$$(9) \qquad p_1 + \cdots + p_r = j - r.$$

Let $p$ represent $p_1 + \cdots + p_r$. If, in (1), the $a_i$ are all multiplied by a number $m$, the values of $y_j(0)$ are multiplied by $m^{p-j}$. If $|m| > 1$, each $y_j(0)$ with $j > q$ will be multiplied by a number of modulus not greater than unity.

We consider a $y(x)$, (with definite $a_i$), which satisfies (3) for some very small $\epsilon$. Let $m$ be such that the greatest of the quantities $|ma_i|$, $i = 1, \cdots, r$, has unity for modulus. Then, by what follows (3), $|m| > 1$. Let

$$\bar{y}(x) = \prod_{i=1}^{r} (x - ma_i)^{p_i}.$$

We inspect the relation (8) as formed for $\bar{y}$. First we let $j = q + r$. In that case, (9) cannot hold. Every $|\bar{y}_i(0)|$ with $q < i \leqq q + r$ is small. Furthermore, because $|ma_i| \leqq 1$, $i = 1, \cdots, r$, there are bounds, independent of $\epsilon$, for the values of the coefficients in (8) at $x = 0$. We infer that $|\bar{y}_q(0)|$ is small. Now, supposing that $q > 0$, let $j = q + r - 1$. We find from (8) that $|\bar{y}_{q-1}(0)|$ is small. Continuing, we find that every $|\bar{y}_i(0)|$ with $i \leqq q + r$ is small.

Let $g$ be such that $|ma_g| = 1$. Then the function

$$(10) \qquad \bar{y}(x)/(x - ma_g)^{p_g}$$

is small, together with its first $q + r$ derivatives, for $x = 0$. It is clear that we can use a single $g$ and obtain a sequence of functions (10) which is such that the values at $x = 0$ of the $k$th function of the sequence and its first $q + r$ derivatives tend toward zero as $k$ increases. By the case of $n = r - 1$, this is impossible. The theorem is proved.

4. **Attainment of bound.** We prove, for $n > 1$, the theorem:

THEOREM. *If no proper subset of the $p_i$ has an integral sum, the family* (1) *has, for every $x$, contact with $y = 0$ of order precisely $q + n - 1$.*

It suffices to show that, when the $p_i$ satisfy the hypothesis, there are values of the $a_i$ distinct from zero such that $y_j(0) = 0$, $j = q + 1, \cdots, q + n - 1$. Such $a_i$ being found, we can multiply them by a small $m$ distinct from zero and obtain a function (1) which is small, for $x = 0$, together with its first $q + n - 1$ derivatives.

The existence of $a_i$ as just described will be established if we can prove that there are numbers $b_1, \cdots, b_n$, distinct from zero, such that the function

$$z = \prod_{i=1}^{n} (1 + b_i x)^{p_i}$$

has derivatives, from the $(q+1)$st to the $(q+n-1)$st inclusive which vanish for $x=0$. The $n-1$ derivatives in question, which we represent by $Z_{q+1}, \cdots, Z_{q+n-1}$, are homogeneous polynomials in the $n$ letters $b_i$. When the $Z_q$ are equated to zero, they determine a non-vacuous algebraic manifold each of whose essential irreducible components is of dimension not less than unity.[5] Thus there is at least one set of numbers $b_1, \cdots, b_n$ which annul the $Z_j$ and are not all zero. We assume in what follows that there is such a set in which the $b_i$ are not all distinct from zero, and prove that some proper subset of the $p_i$ has an integral sum.

We may now work under the assumption that, for some integer $t$ with $0<t<n$, there exist numbers $c_1, \cdots, c_t$, all distinct from zero, such that the function

$$u = \prod_{i=1}^{t} (1 + c_i x)^{p_i}$$

has derivatives from the $(q+1)$st to the $(q+n-1)$st inclusive which vanish for $x=0$. If we put $d_i = -1/c_i$, we find that the function

$$v = \prod_{i=1}^{t} (x - d_i)^{p_i} \tag{11}$$

has derivatives from order $q+1$ through order $q+n-1$ which vanish for $x=0$. For the derivatives $v_j$ of $v$, there exists a relation, analogous to (7), which expresses each $v_j$ in terms of the derivatives which precede it if $j \leqq t$, and in terms of the $t$ derivatives which precede it if $j>t$. In this relation, the coefficient of $v_j$ is $(-1)^t d_1 \cdots d_t$ when $x=0$. Thus, as $v_{q+1}, \cdots, v_{q+n-1}$ vanish for $x=0$, and as they include the $t$ derivatives which precede $v_{q+n}$, $v_{q+n}$ and, then, all the derivatives which follow it, vanish for $x=0$. In other words, $v$ is a polynomial. Thus $p_1 + \cdots + p_t$ is integral and the theorem is proved.

When the $p_i$ are not all integers, $Z_{q+1}$ consists of at least two terms. It is then possible to annul $Z_{q+1}$ with $b_i$ which are all distinct from zero, so that, by what precedes, the order of contact is at least $q+1$. In particular, when $n=2$, the order of contact is $q+1$ except when $p_1$ and $p_2$ are both integers.

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[5] van der Waerden, *Einführung in die algebraische Geometrie*, Berlin, 1939, §41.