THE RADICAL OF A NON-ASSOCIATIVE ALGEBRA

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1. **Introduction.** An algebra $\mathfrak{A}$ is said to be nilpotent of index r if every product of r quantities of $\mathfrak{A}$ is zero, and is said to be a zero algebra if it is nilpotent of index two. It is said to be simple if it is not a zero algebra and its only nonzero (two-sided) ideal is itself, and is said to be semi-simple if it is a direct sum of simple algebras.

The radical of an associative algebra $\mathfrak{A}$ is a nilpotent ideal $\mathfrak{N}$ of $\mathfrak{A}$ which is maximal in the strong sense in that it contains[1] all nilpotent ideals of $\mathfrak{A}$. No such ideal exists in an arbitrary non-associative algebra, and so the radical of such an algebra has never[2] been defined. However the property that $\mathfrak{A}-\mathfrak{N}$ be semi-simple is really the vital one and we shall define the concept of radical here by proving this theorem.

THEOREM 1. *Every algebra $\mathfrak{A}$ which is homomorphic to a semi-simple algebra has an ideal $\mathfrak{N}$, which we shall call the* **radical** *of $\mathfrak{A}$, such that $\mathfrak{A}-\mathfrak{N}$ is semi-simple, $\mathfrak{N}$ is contained in every ideal $\mathfrak{B}$ of $\mathfrak{A}$ for which $\mathfrak{A}-\mathfrak{B}$ is semi-simple.*

The hypothesis that $\mathfrak{A}$ shall be homomorphic to a semi-simple algebra is equivalent to the property that there shall exist an ideal $\mathfrak{B}$ in $\mathfrak{A}$ such that $\mathfrak{A}-\mathfrak{B}$ shall be semi-simple. It is a necessary assumption even in the associative case, since $\mathfrak{A}$ may be nilpotent and then $\mathfrak{A}=\mathfrak{N}$, every $\mathfrak{A}-\mathfrak{B}$ is nilpotent. Moreover it is satisfied by every algebra $\mathfrak{A}$ with a unity quantity. We shall, nevertheless, carry our study a step farther in that we shall define explicitly a certain proper ideal $\mathfrak{N}$ for every algebra $\mathfrak{A}$ such that either $\mathfrak{N}$ is the radical of $\mathfrak{A}$ in the sense above or $\mathfrak{A}$ is not homomorphic to a semi-simple algebra. In the latter case $\mathfrak{A}-\mathfrak{N}$ is a zero algebra.

Our results will be consequences of the remarkable fact[3] that the major structural properties of any non-associative algebra $\mathfrak{A}$ over $\mathfrak{F}$ are determined by almost the same properties of a certain related associative algebra $T(\mathfrak{A})$. We define the right multiplications R_x and the

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[1] For these results see my *Structure of Algebras*, American Mathematical Society Colloquium Publications, vol. 24, 1939, chap. 2.

[2] For the case of alternative algebras see M. Zorn, *Alternative rings and related questions* I: *Existence of the radical*, Annals of Mathematics, (2), vol. 42 (1941), pp. 676–686.

[3] Cf. my *Non-associative algebras* I: *Fundamental concepts and isotopy*, Annals of Mathematics, (2), vol. 43 (1942), pp. 685–708. See also N. Jacobson, *A note on non-associative algebras*, Duke Mathematical Journal, vol. 3 (1937), pp. 544–548.

left multiplications L_x of $\mathfrak{A}$ for every x of $\mathfrak{A}$ to be the respective linear transformations

$$a \to a \cdot x = aR_x, \qquad a \to x \cdot a = aL_x, \qquad\qquad a \text{ in } \mathfrak{A},$$

on $\mathfrak{A}$, and let the transformation algebra $T(\mathfrak{A})$ of $\mathfrak{A}$ be the polynomial ring over $\mathfrak{F}$ generated by the R_x, the L_x, and the identity transformation I. If $\mathfrak{S}$ is any set of linear transformations S on $\mathfrak{A}$ we define $\mathfrak{A}\mathfrak{S}$ to be the linear subspace of $\mathfrak{A}$ spanned by the images aS of every a of $\mathfrak{A}$. Then if $\mathfrak{H}$ is the radical of $T(\mathfrak{A})$ the set $\mathfrak{A}\mathfrak{H}$ is a proper ideal of $\mathfrak{A}$ which is zero if and only if $\mathfrak{H}=0$. When $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra the algebra $T(\mathfrak{A})-\mathfrak{H}$ is a field of order one and we shall prove these theorems.

THEOREM 2. *An algebra $\mathfrak{A}$ is homomorphic to a semi-simple algebra if and only if $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is not a zero algebra.*

THEOREM 3. *If $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra and $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ the algebra $\mathfrak{A}-\mathfrak{B}$ is a zero algebra if and only if $\mathfrak{B}$ contains $\mathfrak{A}\mathfrak{H}$.*

THEOREM 4. *Let $\mathfrak{A}$ be homomorphic to a semi-simple algebra. Then either $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is semi-simple and $\mathfrak{A}\mathfrak{H}$ is the radical of $\mathfrak{A}$ or $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is the direct sum of a semi-simple algebra and a zero algebra $\mathfrak{N}_0=\mathfrak{N}-\mathfrak{A}\mathfrak{H}$ such that $\mathfrak{N}$ is the radical of $\mathfrak{A}$.*

We shall close our discussion with a study of algebras with a unity quantity and the radicals of isotopes with unity quantities. Moreover we shall exhibit an algebra with a unity quantity and a radical which is a field.

2. **A fundamental lemma.** Let $\mathfrak{B}$ be a linear subspace of an algebra $\mathfrak{A}$ of order n over $\mathfrak{F}$ and m be the order of $\mathfrak{B}$ so that there exists an idempotent E of rank m in the algebra $(\mathfrak{F})_n$ of all linear transformations on $\mathfrak{A}$ such that

$$\mathfrak{B} = \mathfrak{A}E.$$

Then $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ if and only if

$$ET(\mathfrak{A}) = ET(\mathfrak{A})E. \tag{1}$$

Since $T(\mathfrak{A})$ contains the identity transformation I it follows that

$$\mathfrak{B}T(\mathfrak{A}) = \mathfrak{B}. \tag{2}$$

We let $\mathfrak{S}$ be the intersection

$$T(\mathfrak{A})E \cap T(\mathfrak{A}), \tag{3}$$

so that $\mathfrak{S}$ consists of all $S=SE$ in $T(\mathfrak{A})$. Then $\mathfrak{S}T(\mathfrak{A})$ is contained in $T(\mathfrak{A})ET(\mathfrak{A})=T(\mathfrak{A})ET(\mathfrak{A})E$ by (1), $\mathfrak{S}T(\mathfrak{A})$ is contained in $\mathfrak{S}$. Also $T(\mathfrak{A})T(\mathfrak{A})E=T(\mathfrak{A})E$, $T(\mathfrak{A})\mathfrak{S}=\mathfrak{S}$ is an ideal[4] of $T(\mathfrak{A})$. Then $T(\mathfrak{A})-\mathfrak{S}$ is defined and we may prove the fundamental lemma.

LEMMA 1. *The algebra* $T(\mathfrak{A}-\mathfrak{B})$ *is equivalent to* $T(\mathfrak{A})-\mathfrak{S}$.

For let a be any quantity of $\mathfrak{A}$ and $\{a\}=a+\mathfrak{B}$ be the corresponding coset in the decomposition of the additive group $\mathfrak{A}$ relative to $\mathfrak{B}$. If T is in $T(\mathfrak{A})$ the set $\mathfrak{B}T$ is contained in $\mathfrak{B}$, the coset $\{aT\}=aT+\mathfrak{B}$ is independent of a. Then the correspondence

$$a \rightarrow \{aT\} = \{a\}T_0 \tag{4}$$

is a transformation T_0 of $\mathfrak{A}-\mathfrak{B}$ uniquely determined for every T of $T(\mathfrak{A})$. Moreover T_0 is a linear transformation. But then we have determined a mapping

$$T \rightarrow T_0 \tag{5}$$

of $T(\mathfrak{A})$ on a set $\mathfrak{T}_0$ of linear transformations T_0 on $\mathfrak{A}-\mathfrak{B}$. It is clear from (4) that

$$\{a(T_1\alpha + T_2\beta)\} = \{aT_1\}\alpha + \{aT_2\}\beta = \{a\}(T_{10}\alpha + T_{20}\beta), \tag{6}$$

$$\{a(T_1T_2)\} = \{(aT_1)T_2\} = \{aT_1\}T_{20} = \{a\}T_{10}T_{20} \tag{7}$$

for every α and β of $\mathfrak{F}$, T_1 and T_2 of $T(\mathfrak{A})$. Then (5) determines a homomorphism of $T(\mathfrak{A})$ on $\mathfrak{T}_0$.

The general right multiplication $R_{\{x\}}$ of $\mathfrak{A}-\mathfrak{B}$ is the transformation $\{a\}\rightarrow\{a\}\cdot\{x\}$, and this is the transformation $(R_x)_0$ given by (5). For

$$\{a\}\cdot\{x\} = \{a\cdot x\} = \{aR_x\}. \tag{8}$$

Similarly $L_{\{x\}}=(L_x)_0$, $\mathfrak{T}_0$ contains $T(\mathfrak{A}-\mathfrak{B})$. If $u_1, \cdots, u_n$ are a basis of $\mathfrak{A}$ over $\mathfrak{F}$ and $S_i=R_{u_i}$, $T_i=L_{u_i}$ every transformation of $T(\mathfrak{A})$ is a polynomial $T=\phi(I, S_1, \cdots, S_n, T_1, \cdots, T_n)$, $T_0=\phi(I, S_{10}, \cdots, S_{n0}, T_{10}, \cdots, T_{n0})$ is in $T(\mathfrak{A}-\mathfrak{B})$, $T(\mathfrak{A}-\mathfrak{B})=T_0$. If $T_0=0$ we have $\{aT\}=0$ for every a, aT is in $\mathfrak{B}$ for every a of $\mathfrak{A}$, $aT=aTE$, $T=TE$ is in $\mathfrak{S}$. Thus the algebra $T(\mathfrak{A})$ is homomorphic under (5) to $T(\mathfrak{A}-\mathfrak{B})$ such that $\mathfrak{S}$ is the ideal of all transformations T of $T(\mathfrak{A})$ such that $T_0=0$. Then $T(\mathfrak{A})-\mathfrak{S}$ is equivalent to $T(\mathfrak{A}-\mathfrak{B})$. This proves our lemma.

3. **Algebras with a semi-simple transformation algebra.** A quantity

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[4] This proof is so brief that I repeat it rather than refer to the proof in the article quoted in Footnote 3.

$z \neq 0$ of an algebra A is called an absolute divisor of zero if $z \cdot a = a \cdot z = 0$ for every a of A. Suppose first that $\mathfrak{A}$ contains no absolute divisor of zero and that $T(\mathfrak{A})$ is semi-simple. Then $T(\mathfrak{A}) = \mathfrak{T}_1 \oplus \cdots \oplus \mathfrak{T}_r$ for simple algebras $\mathfrak{T}_i$ and $\mathfrak{T}_i = T(\mathfrak{A})E_i$ where E_i is the unity quantity of $\mathfrak{T}_i$, E_i is a nonzero idempotent of the center of $T(\mathfrak{A})$. We let $\mathfrak{A}_i = \mathfrak{A}E_i$ and have (1) for each E_i, $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}$. Now $\mathfrak{A}_i = \mathfrak{A}_i E_i$ and it follows that the $\mathfrak{A}_i$ are supplementary ideals $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$. Write $\mathfrak{A} = \mathfrak{A}_i \oplus \mathfrak{B}_i$ and have $T(\mathfrak{A}) = \mathfrak{T}_i \oplus \mathfrak{S}_i$, $\mathfrak{T}_i = T(\mathfrak{A})E_i$, $\mathfrak{S}_i = T(\mathfrak{A})E_{i0}$ where $E_{i0} = I - E_i$. Then $\mathfrak{S}_i$ is the algebra $\mathfrak{S}$ of Lemma 1 for $\mathfrak{B} = \mathfrak{B}_i$, $T(\mathfrak{A} - \mathfrak{B}) \simeq T(\mathfrak{A}) - \mathfrak{S}_i$. Clearly $\mathfrak{A} - \mathfrak{B}_i = \mathfrak{A}_i$, $T(\mathfrak{A}) - \mathfrak{S}_i = \mathfrak{T}_i$, $T(\mathfrak{A}_i) = \mathfrak{T}_i$ is simple. But $\mathfrak{A}_i$ has no absolute divisor of zero and then is known[5] to be simple when $\mathfrak{T}_i$ is simple. Thus we have shown that if $T(\mathfrak{A})$ is semi-simple and $\mathfrak{A}$ has no absolute divisors of zero it is semi-simple.

If $\mathfrak{B}$ is a linear subset of $\mathfrak{A}$ we have $\mathfrak{B}\mathfrak{A} \subset \mathfrak{B}T(\mathfrak{A})$, $\mathfrak{A}\mathfrak{B} \subset \mathfrak{B}T(\mathfrak{A})$. Then if $\mathfrak{H}$ is any right ideal of $T(\mathfrak{A})$ we have $(\mathfrak{A}\mathfrak{H})\mathfrak{A} \subset (\mathfrak{A}\mathfrak{H})T(\mathfrak{A}) \subset \mathfrak{A}\mathfrak{H}$, $\mathfrak{A}(\mathfrak{A}\mathfrak{H}) \subset (\mathfrak{A}\mathfrak{H})T(\mathfrak{A}) \subset \mathfrak{A}\mathfrak{H}$. Hence $\mathfrak{A}\mathfrak{H}$ is an ideal of $\mathfrak{A}$. If $\mathfrak{H} \neq 0$ is a nilpotent ideal of $T(\mathfrak{A})$ we cannot have $\mathfrak{A}\mathfrak{H} = 0$. Also $\mathfrak{A}\mathfrak{H} \neq \mathfrak{A}$ since otherwise $\mathfrak{H}^t = 0$ would imply that $\mathfrak{A} = 0$. Let then $\mathfrak{A}$ be semi-simple, $\mathfrak{H}$ be the radical of $T(\mathfrak{A})$. We write $\mathfrak{A} = \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r$ for simple algebras $\mathfrak{A}_i$ and may choose pairwise orthogonal idempotents E_i of $(\mathfrak{F})_n$ such that $\mathfrak{A}_i = \mathfrak{A}E_i$, $E_1 + \cdots + E_r = I$. Then $\mathfrak{A}_i$ is an ideal of $\mathfrak{A}$, $E_iT(\mathfrak{A}) = E_iT(\mathfrak{A})E_i$, $E_i\mathfrak{H} = E_i\mathfrak{H}E_i = \mathfrak{H}_i$ is clearly a nilpotent ideal of $E_iT(\mathfrak{A})$. But it follows that $\mathfrak{A}\mathfrak{H} = \mathfrak{A}_1\mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{A}_1\mathfrak{H}_r$ for ideals $\mathfrak{A}_i\mathfrak{H}_i$ of $\mathfrak{A}_i$. This is impossible unless each $\mathfrak{H}_i = 0$ since each $\mathfrak{A}_i$ is simple. Thus if $\mathfrak{A}$ is semi-simple so is $T(\mathfrak{A})$.

Suppose finally that $\mathfrak{A}$ does have absolute divisors of zero and let $\mathfrak{N}$ be the set of all absolute divisors of zero of $\mathfrak{A}$. Then clearly $\mathfrak{N}$ is an ideal of $\mathfrak{A}$ which is a zero algebra, $\mathfrak{N} = \mathfrak{N}E$ for an idempotent E of $(\mathfrak{F})_n$. If $\mathfrak{A} = \mathfrak{N}$ we have $T(\mathfrak{A}) = I\mathfrak{F}$ and $T(\mathfrak{A})$ is semi-simple. Otherwise $I - E = E_0$ is an idempotent of $(\mathfrak{F})_n$, $\mathfrak{A} = \mathfrak{A}(E + E_0)$ is the direct sum $\mathfrak{A} = \mathfrak{G} \oplus \mathfrak{N}$ where $\mathfrak{G} = \mathfrak{A}E_0$ contains no absolute divisors of zero. If a and x are in $\mathfrak{A}$ we write $x = g + h$ with g in $\mathfrak{G}$ and h in $\mathfrak{N}$, $a \cdot x = a \cdot g = aR_g$, $R_x = R_g$, $(a \cdot x)E_0$ is in $\mathfrak{G}$, $R_xE_0 = R_x$ for every x of $\mathfrak{A}$. Similarly every L_x is in $T(\mathfrak{A})E_0$ and it is clear that $T(\mathfrak{G})$ is equivalent to $T(\mathfrak{A})E_0$. The algebra $\mathfrak{S}$ of Lemma 1 defined for $\mathfrak{B} = \mathfrak{G}$ is $T(\mathfrak{A})E_0$ and is an ideal of $T(\mathfrak{A})$, $T(\mathfrak{A}) = T(\mathfrak{A})E_0 + I\mathfrak{F}$. The algebra $\mathfrak{S}$ of Lemma 1 defined for $\mathfrak{B} = \mathfrak{N}$ is $T(\mathfrak{A})E$ and is EF since $E_0E = 0$. Then $T(\mathfrak{A}) = T(\mathfrak{A})E_0 \oplus EF$, $T(\mathfrak{A})E_0$ is semi-simple when $T(\mathfrak{A})$ is semi-simple, $T(\mathfrak{G})$ is semi-simple

[5] In the paper referred to in Footnote 3, N. Jacobson defined $T(\mathfrak{A})$ to be generated by the right and left multiplications of $\mathfrak{A}$ and with *I omitted*. He then proved our result. We require the more general statement including the case where $\mathfrak{A}$ may be a zero algebra and so refer to Lemma 10 of my own paper of that reference.

and so is $\mathfrak{G}$. Conversely, if $\mathfrak{G}$ is semi-simple so is $T(\mathfrak{A})E_0$ and so is $T(\mathfrak{A})$. We have proved this lemma.

LEMMA 2. *The transformation algebra* $T(\mathfrak{A})$ *is semi-simple if and only if* $\mathfrak{A}$ *is either semi-simple, a zero algebra, or a direct sum of a semi-simple algebra and a zero algebra.*

4. **The radical of an algebra.** If $\mathfrak{A}$ is an associative semi-simple algebra and $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ we have $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$, where $\mathfrak{C}$ is an ideal of $\mathfrak{A}$ equivalent to $\mathfrak{A}-\mathfrak{B}$ and is semi-simple. Let $\mathfrak{A}$ now be an associative algebra with radical $\mathfrak{N}\neq 0$ and $\mathfrak{B}$ be an ideal of $\mathfrak{A}$. If $\mathfrak{B}$ contains $\mathfrak{N}$ the algebra $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{N})-(\mathfrak{B}-\mathfrak{N})$ and is semi-simple by the argument above. Conversely, let $\mathfrak{B}$ be an ideal of $\mathfrak{A}$ such that $\mathfrak{A}-\mathfrak{B}$ is semi-simple, $\mathfrak{A}-\mathfrak{B}$ contains no properly nilpotent classes. Then every properly nilpotent quantity of $\mathfrak{A}$ must define the zero class of $\mathfrak{A}-\mathfrak{B}$, $\mathfrak{B}$ contains all properly nilpotent quantities of $\mathfrak{A}$, $\mathfrak{B}$ contains $\mathfrak{N}$. This proves Theorem 1 in the associative case.

We now let $\mathfrak{B}$ be any ideal of an arbitrary algebra $\mathfrak{A}$, $\mathfrak{H}$ be the radical of $T(\mathfrak{A})$ so that $\mathfrak{A}\mathfrak{H}$ is a proper ideal of $\mathfrak{A}$. If $\mathfrak{A}-\mathfrak{B}$ is semi-simple so is $T(\mathfrak{A}-\mathfrak{B})$ by Lemma 2, and so is $T(\mathfrak{A})-\mathfrak{S}$ by Lemma 1. But by the result just proved $\mathfrak{S}$ contains $\mathfrak{H}$. However $\mathfrak{B}=\mathfrak{A}E$, $\mathfrak{S}=T(\mathfrak{A})E=\mathfrak{S}E$, $\mathfrak{H}=\mathfrak{H}E$, $\mathfrak{A}\mathfrak{H}=\mathfrak{A}\mathfrak{H}E$ is contained in $\mathfrak{A}\mathfrak{S}=\mathfrak{A}\mathfrak{S}E$ and hence in $\mathfrak{B}$. Then we have proved that the radical $\mathfrak{N}$ of $\mathfrak{A}$ contains $\mathfrak{A}\mathfrak{H}$, $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{A}\mathfrak{H})-(\mathfrak{B}-\mathfrak{A}\mathfrak{H})$. Hence $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ cannot be a zero algebra. If $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is semi-simple our definition implies that $\mathfrak{A}\mathfrak{H}=\mathfrak{N}$. Otherwise $\mathfrak{A}_0=\mathfrak{A}-\mathfrak{A}\mathfrak{H}=\mathfrak{G}\oplus\mathfrak{N}_0$ where $\mathfrak{G}$ is semi-simple and $\mathfrak{N}_0$ is a zero algebra, $\mathfrak{B}-\mathfrak{A}\mathfrak{H}$ is an ideal $\mathfrak{B}_0$ of $\mathfrak{A}_0$ such that $\mathfrak{A}_0-\mathfrak{B}_0$ is semi-simple. If there is a quantity of $\mathfrak{N}_0$ not in $\mathfrak{B}_0$ the corresponding class of $\mathfrak{A}_0-\mathfrak{B}_0$ is an absolute divisor of zero of that algebra, contrary to our hypothesis. Hence $\mathfrak{B}_0$ contains $\mathfrak{N}_0$, $\mathfrak{B}$ contains the algebra $\mathfrak{N}$ of all the quantities in the classes of $\mathfrak{N}_0$, $\mathfrak{N}$ is the radical of $\mathfrak{A}$. This proves Theorem 1 and Theorem 4.

If $\mathfrak{A}$ is homomorphic to an algebra $\mathfrak{A}_0$ and $\mathfrak{B}$ is the set of all quantities of $\mathfrak{A}$ mapped into zero by the given homomorphism then $\mathfrak{A}-\mathfrak{B}$ is equivalent to $\mathfrak{A}_0$. If $\mathfrak{A}-\mathfrak{B}$ is semi-simple we have seen that $\mathfrak{B}$ contains $\mathfrak{A}\mathfrak{H}$, $\mathfrak{A}-\mathfrak{B}$ is equivalent to $(\mathfrak{A}-\mathfrak{A}\mathfrak{H})-(\mathfrak{B}-\mathfrak{A}\mathfrak{H})$, $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ cannot be a zero algebra. This proves Theorem 2. We have also seen that if $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ is a zero algebra and $\mathfrak{A}-\mathfrak{B}$ is a zero algebra then $T(\mathfrak{A}-\mathfrak{B})=T(\mathfrak{A})-\mathfrak{S}$ for an ideal $\mathfrak{S}$ of $T(\mathfrak{A})$. But then by Lemma 2 $T(\mathfrak{A})-\mathfrak{S}$ is semi-simple, $\mathfrak{S}$ contains $\mathfrak{H}$, $\mathfrak{A}\mathfrak{S}=\mathfrak{A}\mathfrak{S}E$ contains $\mathfrak{A}\mathfrak{H}$ and is contained in $\mathfrak{A}E=\mathfrak{B}$. This proves Theorem 3.

5. **The radicals of isotopic algebras.** Let $\mathfrak{A}$ and $\mathfrak{A}_1$ be algebras of the same order so that we may regard them as having quantities in the

same linear space. Then $\mathfrak{A}$ and $\mathfrak{A}_1$ are principal isotopes if multiplication in $\mathfrak{A}_1$ is given by $[a, x]=aR_x^{(1)}$ for $R_x^{(1)}=PR_{xQ}$, P and Q nonsingular linear transformations on $\mathfrak{A}$. If $\mathfrak{A}$ and $\mathfrak{A}_1$ have unity quantities the transformations P and Q are in $T(\mathfrak{A})$ and $T(\mathfrak{A})=T(\mathfrak{A}_1)$.

Let $\mathfrak{B}$ be an ideal of $\mathfrak{A}$, $\mathfrak{B}=\mathfrak{A}E$, $ET(\mathfrak{A})=ET(\mathfrak{A})E$. Then $EP=EPE$, $EQ=EQE$ and $EPP^{-1}=E=(EP)(EP^{-1})$, $P_1=EP$ is a nonsingular quantity of $ET(\mathfrak{A})$. Similarly $Q_1=EQ$ is a nonsingular quantity of $ET(\mathfrak{A})$. Write $\mathfrak{B}_1=\mathfrak{A}_1E$, so that since $ET(\mathfrak{A}_1)=ET(\mathfrak{A}_1)E$ the space $\mathfrak{B}_1$ is an ideal[6] of $\mathfrak{A}_1$. Then if b and y are in $\mathfrak{B}_1$ we have $b=bE$, $y=yE$,

$$(9) \qquad [b, y] = bEPR_{yEQ} = bP_1R_{yQ_1}.$$

It follows that $\mathfrak{B}$ and $\mathfrak{B}_1$ are principal isotopes with isotopy given by

$$(10) \qquad R_y^{(1)} = P_1R_{yQ_1}.$$

Every isotope $\mathfrak{A}_1$ of $\mathfrak{A}$ is equivalent to a principal isotope and we have proved the first part of this theorem.

THEOREM 5. *Let $\mathfrak{A}$ and $\mathfrak{A}_1$ be isotopic algebras with unity quantities. Then every ideal $\mathfrak{B}$ of $\mathfrak{A}$ is an isotope of an ideal $\mathfrak{B}_1$ of $\mathfrak{A}_1$ such that the difference algebras $\mathfrak{A}-\mathfrak{B}$ and $\mathfrak{A}_1-\mathfrak{B}_1$ are isotopes.*

We now observe that the homomorphism (5) of $T(\mathfrak{A})$ on $T(\mathfrak{A}-\mathfrak{B})$ carries every nonsingular P of $T(\mathfrak{A})$ into a nonsingular P_0 of $T(\mathfrak{A}-\mathfrak{B})$. Then if we define

$$(11) \qquad (R_{\{x\}})^{(1)} = P_0R_{\{x\}Q_0},$$

the algebra with multiplication defined by

$$(12) \qquad [\{a\}, \{x\}] = \{a\}(R_{\{x\}})^{(1)}$$

is a principal isotope of $\mathfrak{A}-\mathfrak{B}$. But the difference algebra $\mathfrak{A}_1-\mathfrak{B}_1$ has multiplication defined by $[\{a\}, \{x\}]=\{[a, x]\}=\{aR_x^{(1)}\}=\{aPR_{xQ}\}$ $=\{aP\}R_{\{xQ\}}=\{a\}(R_{\{x\}})^{(1)}$ since $\{aP\}=\{a\}P_0$, $\{xQ\}=\{x\}Q_0$. This proves our theorem.

We should observe that while P_0 and Q_0 are in $T(\mathfrak{A}-\mathfrak{B})$ the transformations P_1 and Q_1 defining the isotopy of B and B_1 need not be in $T(\mathfrak{B})$. This is an evident consequence of the fact that if $\mathfrak{A}$ has a unity quantity so does $\mathfrak{A}-\mathfrak{B}$, but certainly $\mathfrak{B}$ need not have a unity quantity. Observe also that if $\mathfrak{A}$ of order n does not have a unity quantity and we pass to an algebra $\underline{\mathfrak{A}}$ of order $n+1$ with a unity quantity the algebra $\mathfrak{A}$ will be an ideal of $\underline{\mathfrak{A}}$. The results above then become of par-

[6] It follows from this that if $\mathfrak{B}$ is an ideal of $\mathfrak{A}$ the same linear space is an ideal $\mathfrak{B}_1$ of $\mathfrak{A}_1$. However, we wish to prove the stronger result that $\mathfrak{B}$ and $\mathfrak{B}_1$ are isotopic.

ticular importance in the study of isotopes of algebras *without* unity quantities.

We conclude our general results by proving the following theorem

THEOREM 6. *Let* $\mathfrak{A}$ *be an algebra with a unity quantity,* $\mathfrak{H}$ *be the radical of* $T(\mathfrak{A})$. *Then* $\mathfrak{A}\mathfrak{H}$ *is the radical of* $\mathfrak{A}$ *and is isotopic to the radical* $\mathfrak{A}_1\mathfrak{H}_1$ *of any isotope* $\mathfrak{A}_1$ *of* $\mathfrak{A}$ *with a unity quantity. Moreover the semi-simple algebras* $\mathfrak{A}-\mathfrak{A}\mathfrak{H}$ *and* $\mathfrak{A}_1-\mathfrak{A}_1\mathfrak{H}$ *are isotopic.*

For every homomorph $\mathfrak{A}-\mathfrak{B}$ of an algebra $\mathfrak{A}$ with a unity quantity has a unity quantity and cannot be a direct sum of a zero algebra and another algebra. Thus Theorem 4 implies that $\mathfrak{A}\mathfrak{H}$ is the radical of $\mathfrak{A}$. Our result follows from Theorem 5.

6. **An algebra whose radical is a field.** Let $\mathfrak{A}$ be an algebra with a basis e, u, v over $\mathfrak{F}$ so that every quantity of $\mathfrak{A}$ is uniquely expressible in the form $a=\alpha e+\beta u+\gamma v$ for α, β, γ in $\mathfrak{F}$. We let e be the unity quantity of $\mathfrak{A}$ and complete the definition of $\mathfrak{A}$ with the relations

$$u^2 = e, \qquad uv = v, \qquad v^2 = v, \qquad vu = 0.$$

Let $\mathfrak{B}$ be a nonzero ideal of $\mathfrak{A}$ and $a\neq 0$ be in $\mathfrak{B}$ so that the corresponding α, β, γ are not all zero. Then $au=\alpha u+\beta e$, $(au)u=\alpha e+\beta u$, $a-(au)u=\gamma v$, $v[(au)u]=\alpha v$, $v(au)=\beta v$ are all in $\mathfrak{B}$, $\mathfrak{B}$ contains the algebra $\mathfrak{N}$ of order one over $\mathfrak{F}$ spanned by v. Now $(\alpha e+\beta u+\gamma v)v =(\alpha+\beta+\gamma)v$, $v(\alpha e+\beta u+\gamma v)=(\alpha+\gamma)v$, $\mathfrak{N}$ is an ideal of $\mathfrak{A}$. If $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$ for ideals $\mathfrak{B}$ and $\mathfrak{C}$ we have proved that both $\mathfrak{B}$ and $\mathfrak{C}$ would contain $\mathfrak{N}$. This is impossible. Also $\mathfrak{N}$ is a nonzero proper ideal of $\mathfrak{A}$ and $\mathfrak{A}$ cannot be simple. It follows that $\mathfrak{A}$ is not semi-simple. But $\mathfrak{A}=\mathfrak{N}+\mathfrak{G}$ where $\mathfrak{G}$ is the semi-simple associative algebra spanned by e and u, $\mathfrak{A}-\mathfrak{N}=\mathfrak{G}$, $\mathfrak{A}-\mathfrak{N}$ is semi-simple. Then $\mathfrak{N}$ is the radical of $\mathfrak{A}$ according to our definition and is a field of order one over $\mathfrak{F}$.

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