all integral functions $f(z)$ satisfying the conditions $f(t)\in L_1$, $\mathfrak{H}f\in L_1$, $|f(z)|<K_{f,\epsilon}\exp\{(2\alpha+\epsilon)|z|\}$. The proof is based upon a result due to Plancherel and Pólya.[12]

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[12] Commentarii Mathematici Helvetici, vol. 10 (1937–1938), pp. 110–163, §27.

THE BEHAVIOR OF CERTAIN STIELTJES CONTINUED FRACTIONS NEAR THE SINGULAR LINE

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1. **Introduction.** We consider here continued fractions of the form[1]

$$f(z) = \frac{g_0}{1} + \frac{g_1 z}{1} + \frac{(1-g_1)g_2 z}{1} + \frac{(1-g_2)g_3 z}{1} + \cdots, \tag{1.1}$$

in which $g_0\geqq 0$, $0\leqq g_n\leqq 1$, $(n=1, 2, 3, \cdots)$, it being agreed that the continued fraction shall terminate in case some partial numerator vanishes identically. There exists a monotone non-decreasing function $\phi(u)$, $0\leqq u\leqq 1$, such that

$$f(z) = \int_0^1 \frac{d\phi(u)}{1+zu}; \tag{1.2}$$

and, conversely, every integral of this form is representable by such a continued fraction. Put $M(f)=\text{l.u.b.}_{|z|<1}|f(z)|$. Then $M(f)\leqq 1$ if and only if the continued fraction can be written in the form

$$f(z) = \frac{h_1}{1} + \frac{(1-h_1)h_2 z}{1} + \frac{(1-h_2)h_3 z}{1} + \cdots, \tag{1.3}$$

in which $0\leqq h_n\leqq 1$, $(n=1, 2, 3, \cdots)$. These functions are analytic in the interior of the z-plane cut along the real axis from $z=-1$ to $z=-\infty$.

The principal object of this paper is to prove the following theorem:

THEOREM 1.1. *If $0<h_n<1$, $(n=1, 2, 3, \cdots)$, and $h_n\to 1/2$ in such a way that the series $\sum|h_n-1/2|$ converges, then the function $f(z)$ given*

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[1] H. S. Wall, *Continued fractions and totally monotone sequences*, Transactions of this Society, vol. 48 (1940), pp. 165–184.

by (1.3) *approaches a finite limit* $\alpha(s)$ *as* $z\to -s$, $s \geqq 1$, *from the upper half-plane, and the limit* $\overline{\alpha(s)}$, *the complex conjugate of* $\alpha(s)$, *as* $z\to -s$ *from the lower half-plane. The function* $\alpha(s)$ *is continuous, and is real if and only if* $s=1$. *There is a constant* C *such that* $|f(z)| < C$ *over the entire plane of* z *exterior to the cut along the real axis from* $z=-1$ *to* $z=-\infty$.

Inasmuch as the function (1.1) can be written in the form $f(z)=g_0/[1+zf^*(z)]$, where $f^*(z)$ has the form (1.3), one may conclude at once that if $g_0>0$, $0<g_n<1$, $(n=1, 2, 3, \cdots)$, $\sum |g_n-1/2|$ converges, then the function $f(z)$ given by (1.1) approaches a finite limit $\beta(s)$ as $z\to -s$, $s>1$, from the upper half-plane, and the limit $\overline{\beta(s)}$ as $z\to -s$ from the lower half-plane. The function $\beta(s)$ is continuous and not real for $s>1$. The function $f(z)$ given by (1.1) may become infinite as $z\to -1$, for example, if $g_0=1$, $g_n=1/2$ $(n=1, 2, 3, \cdots)$, then $f(z)=(1+z)^{-1/2}$.

2. **Proof of Theorem** 1.1. There is a one to one correspondence between functions of the form (1.3) and functions $e(x)$ which are real when x is real, analytic for $|x|<1$, and for which $M(e)\leqq 1$, such that if $f(z)\leftrightarrow e(x)$ then[2]

$$(2.1)\qquad \frac{1}{2}(1-x)\frac{1-e(x)}{1+xe(x)} = f(z), \quad z = 4x/(1-x)^2, \quad |x| < 1.$$

(i) The transformation $z=4x/(1-x)^2$ maps the interior of the circle $|x|=1$ one to one upon the interior of the z-plane cut along the real axis from $z=-1$ to $z=-\infty$. Hence it follows at once from (2.1) that if $M(e)<1$, then

$$|f(z)| \leqq \frac{1+M(e)}{1-M(e)} = C,$$

over the entire domain of analyticity of $f(z)$.

(ii) In (2.1) put $x=\xi+i\eta$, $e(x)=u+iv$, $f(z)=P+iQ$, where ξ, η, u, v, P, Q are all real. We then find for Q the value

$$(2.2)\qquad Q = \frac{\eta(u^2+v^2-1)+v(\xi^2+\eta^2-1)}{2|1+xe(x)|^2}.$$

If $s\geqq 1$, $\sigma=[s-2+2i(s-1)^{1/2}]/s$, so that $|\sigma|=1$, then as $x\to\sigma$ from the interior of the circle $|x|=1$, z must approach $-s$ from the upper half-plane. If $M(e)<1$, and $e(x)$ approaches a limit $e(\sigma)$ as $x\to\sigma$, $|x|<1$, then it follows from (2.1) that $f(z)$ approaches a finite limit

[2] H. S. Wall, *Some recent developments in the theory of continued fractions*, this Bulletin, vol. 47 (1941), pp. 405–423; Theorem 5.1, p. 415.

$\alpha(s)$ as $z\to -s$ from the upper half-plane; and from (2.2) it follows that Q has the limit

$$\frac{(s-1)^{1/2}}{s}\cdot\frac{|e(\sigma)|^2-1}{|1+\sigma e(\sigma)|^2}, \tag{2.3}$$

where is zero if and only if $s=1$. Hence $\alpha(s)$ is real if and only if $s=1$. Inasmuch as $f(\bar z)=\overline{f(z)}$, it follows that $f(z)$ has the limit $\overline{\alpha(s)}$ as $z\to -s$ from the lower half-plane. Clearly $\alpha(s)$ is continuous if $e(x)$ is continuous for $|x|\leqq 1$.

(iii) To complete the proof of Theorem 1.1 it remains to be proved that when $\sum|h_n-1/2|$ converges then $M(e)<1$ and $e(x)$ is continuous for $|x|\leqq 1$. Put $e_0(x)=e(x)$,

$$e_{n+1}(x)=\frac{1}{x}\,\frac{t_n-e_n(x)}{1-t_ne_n(x)},\qquad t_n=e_n(0);\ n=0,1,2,\cdots. \tag{2.4}$$

Then $t_{n-1}=1-2h_n$ $(n=1,2,3,\cdots)$. Now, Schur[3] proved that if $|t_{n-1}|<1$, $(n=1,2,3,\cdots)$, and $\sum|t_n|$ is convergent, then $M(e)<1$, and $e(x)$ is continuous for $|x|\leqq 1$. Since $0<h_n<1$ by hypothesis, it follows that $-1<t_{n-1}<1$; and since the series $\sum|h_n-1/2|$ converges by hypothesis, it follows that $\sum|t_n|$ converges.

This completes the proof of Theorem 1.1.

It will be seen from (2.3) that if $f(z)$ has a real limit as $z\to -s$, $s>1$, then $M(e)=1$. This is true also if $f(z)$ becomes infinite as $z\to -s$, $s\geqq 1$, and in this case $e(x)\to -1/\sigma$ as $x\to\sigma$. Inasmuch as $\lim_{z\to -s}(z+s)f(z)=0$, $\lim_{z\to\infty}f(z)=0$, if $M(e)<1$, it follows that the corresponding mass function $\phi(u)$ (cf. (1.2)), is continuous for $0\leqq u\leqq 1$ in this case.[4]

3. **An example.** If we apply the transformation (2.4) to a function $f(z)$ of the form (1.3) we obtain a sequence of functions $f_0(z)=f(z)$, $f_1(z)$, $f_2(z)$, $\cdots$ all having continued fraction expansions of the same character as that of $f(z)$. Suppose that in (1.3), $0<h_n<1$, $(n=1,2,3,\cdots)$, and that the series $\sum|h_n-1/2|$ converges. On applying Theorem 1.1 we find at once that as $z\to -s$, $s\geqq 1$, $I(z)>0$:

$$\lim f_1(z)=-\frac{1}{s}\,\frac{g_1-\alpha(s)}{1-g_1\alpha(s)}=\alpha_1(s);$$

and that $\alpha_1(s)$ is real if and only if $s=1$; $\alpha_1(1)=1$; $\alpha_1(s)$ is continuous for $s\geqq 1$. By mathematical induction, $f_2(z)$, $f_3(z)$, $\cdots$ also have this

[3] I. Schur, *Über Potenzreihen, die im Innern des Einheitskrieses beschränkt sind*, Journal für die reine und angewandte Mathematik, vol. 147 (1916), pp. 205–232, and vol. 148 (1917), pp. 122–145.

[4] I. J. Schoenberg, *Über die asymptotische Verteilung reeller Zahlen* mod 1, Mathematische Zeitschrift, vol. 28 (1928), pp. 171–199; p. 179.

property. Let $h_1'/1+(1-h_1')h_2'z/1+(1-h_2')h_3'z/1+\cdots$ be the continued fraction for $f_1(z)$. *Then we shall prove that* $\sum|h_n'-1/2|$ *may diverge although* $\sum|h_n-1/2|$ *converges, and that the convergence of the series* $\sum|h_n-1/2|$ *is not necessary in order that the conclusion in Theorem* 1.1 *shall hold.* For this purpose, let $h_n=1/2$, $(n=1, 2, 3, \cdots)$. Then $f(z)=1/[1+(1+z)^{1/2}]$, $(f(0)=1/2)$, and $f_1(z)=1/[1+(1+z)^{1/2}][1+2(1+z)^{1/2}]$, $(f_1(0)=1/6)$. The function $f_1(z)$ has the properties stated in Theorem 1.1 for the function $f(z)$ of that theorem, excepting that, as we shall see, the series $\sum|h_n'-1/2|$ diverges. In fact, $h_{2n}'=(4n+3)/2(4n+1)$, $h_{2n-1}'=(4n-3)/2(4n-1)$, $(n=1, 2, 3, \cdots)$, in consequence of the following theorem:

THEOREM 3.1. *Let k be a parameter subject only to the conditions*

$$k \neq (3-4n)/2, \qquad (1-4n)/2, \qquad n=1, 2, 3, \cdots, \tag{3.1}$$

and put

$$h_{2n-1}^{(k)} = (4n-3)/2(4n-3+2k),$$
$$h_{2n}^{(k)} = (4n-1+4k)/2(4n-1+2k), \qquad n=1, 2, 3, \cdots.$$

Then the continued fraction $f_k(z)=h_1^{(k)}/1+(1-h_1^{(k)})h_2^{(k)}z/1+(1-h_2^{(k)})h_3^{(k)}z/1+\cdots$ *converges uniformly in a sufficiently small neighborhood of $z=0$, and the analytic function $f_k(z)$ satisfies the relation*

$$f_{k+1}(z) = \frac{1}{z}\,\frac{h_1^{(k)}-f_k(z)}{1-h_1^{(k)}f_k(z)}. \tag{3.2}$$

PROOF. The uniform convergence follows from the fact that all the partial numerators after the first are numerically less than or equal to $1/4$ for z in a sufficiently small neighborhood of the origin. To prove (3.2), write the right-hand member in the form:

$$\frac{1}{z}\left\{h_1^{(k)} - \frac{1-(h_1^{(k)})^2}{-h_1^{(k)}+1/f_k(z)}\right\} = \frac{1}{z}\left\{h_1^{(k)} - \frac{h_1^{(k)}}{1} + \frac{h_2^{(k)}z/(1+h_1^{(k)})}{1} + \frac{(1-h_2^{(k)})h_3^{(k)}z}{1} + \frac{(1-h_3^{(k)})h_4^{(k)}z}{1} + \cdots\right\}.$$

We are to show that this is equal to $f_{k+1}(z)$. This can be done by showing that the odd part of the last continued fraction is identical with the even part of the continued fraction for $f_{k+1}(z)$. We omit here the details of the calculation.[5]

Let $e_n(x) \leftrightarrow f_n(z)$, $(n=0, 1, 2, \cdots)$. Then we find for the e_n's the following recursion formulas:

$$(3.3) \qquad e_{n+1}(x) = \frac{1}{x} \frac{k_n + (2 - k_n)x + (3x - 1)e_n(x)}{(3 - x) + (2 - k_n + k_n x)e_n(x)}, \quad k_n = e_n(0),$$

$(n=0, 1, 2, \cdots)$. For the special example under consideration, $e(x) = e_0(x) \equiv 0$ and $e_1(x) = 2/(3-x)$. Hence, although $M(e) < 1$ in this case, nevertheless $M(e_1) = 1$. From the way in which (3.3) was obtained it follows that if $e_0(x)$ is an arbitrary function which is real when x is real, analytic for $|x| < 1$, and such that $M(e_0) \leqq 1$, then the functions $e_1(x)$, $e_2(x)$, $\cdots$ are all of this same character.

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[5] O. Perron, *Die Lehre von den Kettenbrüchen*, 2d edition, Leipzig and Berlin, 1929 p. 201.